The Foundations of Arithmetic

GOTTLOB FREGE
1848–1925

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Die Grundlagen der Arithmetik

Translated from the German with AI (Claude) and edited



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INTRODUCTION

Gottlob Frege (1848–1925) taught mathematics at the University of Jena for more than forty years and was little read in his lifetime. In 1879 he had invented, in his *Begriffsschrift*, the first modern system of logic. *The Foundations of Arithmetic* (1884) sets out, almost without symbols, what he meant to do with it: to show that arithmetic is a branch of logic, and that numbers are neither physical things nor ideas in anyone's mind.

The book begins by asking what the number one is, and finds that mathematicians cannot say. Frege then takes apart the answers of his predecessors—Mill's view that arithmetic is an empirical science, Kant's that its truths are synthetic a priori, and the many accounts of number as a heap of things, a property of things, or a mental image. His own answer rests on three principles stated in the Introduction, the second of which is famous: a word has meaning only in the context of a sentence. A statement of number, he argues, says something about a concept; the number of a concept is defined through the idea of one-to-one correlation, and the series of natural numbers is built up from zero by logic alone.

Bertrand Russell later showed that the system in which Frege tried to carry out this program was inconsistent. But the *Foundations* has outlived that failure. It is a classic of philosophical prose and a founding text of analytic philosophy and the philosophy of mathematics.

Introduction

Einleitung

What is the number one? Three guiding principles; then §§ 1–4: the demand for rigour, and the questions whether the laws of number are analytic or synthetic, a priori or a posteriori.

E.1 If we ask what the number one is, or what the sign 1 means, we will usually get the answer: well, a thing. And if we then point out that the sentence "the number one is a thing" is not a definition, because it has the definite article on one side and the indefinite on the other, and that it says only that the number one belongs among things, but not which thing it is, we may be invited to choose any thing we like and call it one. But if everyone had the right to understand by this name whatever he pleased, then the same sentence about one would mean different things for different people; such sentences would have no common content. Some may dismiss the question by pointing out that the meaning of the letter a in arithmetic cannot be specified either, and that if one says "a means a number", the same fault can be found in this as in the definition "one is a thing". Now, dismissing the question is entirely justified in the case of a: it does not mean a particular, specifiable number, but serves to express the generality of propositions. If in $a + a - a = a$ we put for a any number we like, but the same one throughout, we always obtain a true equation. That is the sense in which the letter a is used. But with one the matter is essentially different. In the equation $1 + 1 = 2$, can we put the same object for 1 both times, the moon, say? It seems, rather, that we must put something different for the first 1 than for the second. Why must pre-

cisely what would be a mistake in the other case happen here? Arithmetic cannot make do with the letter a alone. It must also use other letters, b, c, etc., to express relations between different numbers in general. So, one would think, the sign 1 could not suffice either, if it served in a similar way to lend generality to propositions. But does the number one not appear as a definite object with specifiable properties, for example that of remaining unchanged when multiplied by itself? In this sense no properties of a can be specified; for what is asserted of a is a property common to numbers, whereas $1^1 = 1$ asserts nothing of the moon, nor of the sun, nor of the Sahara, nor of the Peak of Tenerife; for what could be the sense of such an assertion?

E.2 Even most mathematicians will probably have no adequate answer ready to such questions. Is it not shameful for science to be so unclear about its most immediate and seemingly so simple object? All the less will anyone be able to say what number is. When a concept that lies at the foundation of a great science presents difficulties, it is surely an inescapable task to investigate it more closely and to overcome those difficulties. This is all the more so because it will hardly be possible to achieve full clarity about negative, fractional and complex numbers so long as our insight into the foundation of the whole edifice of arithmetic is still deficient.

E.3 Many, of course, will not think this worth the trouble. This concept, they believe, is adequately treated in the elementary textbooks and thereby settled for life. Who imagines there is anything left to learn about so simple a matter! The concept of positive whole number is thought to be so free of every difficulty that it can be treated exhaustively and scientifically for children, and that everyone knows all about it precisely, without further reflection and without any acquaintance with what others have thought. And so that first precondition of learning is often missing: the knowledge of not knowing. The result is that people still content themselves with a crude conception,

although *Herbart*[1] long ago taught a more correct one.[2] It is saddening and discouraging that in this way an insight already won keeps threatening to be lost again. Many a piece of work seems to come to nothing because people, in their imagined riches, believe they have no need to make its fruits their own. This work too, I am well aware, is exposed to that danger. I meet that crudeness of conception when calculation is called aggregative, mechanical thinking[3]. I doubt that any such thinking exists at all. Aggregative ideation might more readily be allowed; but it is of no significance for calculation. Thinking is essentially the same everywhere: different kinds of laws of thought do not come into play depending on the object. The differences consist only in greater or lesser purity and independence from psychological influences and from external aids to thinking such as language, numerals and the like, and then perhaps also in the fineness of the structure of the concepts. But precisely in this last respect mathematics is hardly surpassed by any science, not even by philosophy.

E.4 This work will show that even an apparently peculiarly mathematical inference, such as that from n to n + 1, rests on the general laws of logic, and that no special laws of aggregative thinking are needed. One can, of course, use numerals mechanically, just as one can talk like a parrot; but that could hardly be called thinking. It is possible only after actual thinking has so developed the mathematical sign language that it does the thinking for us, as people say. This does not prove that numbers are formed in some peculiarly mechanical way, as heaps of sand are formed from grains of quartz. It is in the interest of mathem-

1 FREGE'S NOTE. Sämmtliche Werke, ed. Hartenstein, vol. X, pt 1, Umriss pädagogischer Vorlesungen [Outline of Lectures on Education], § 252, note 2: "Two does not mean two things, but doubling," etc.

2 Johann Friedrich Herbart (1776–1841), German philosopher, psychologist and theorist of education.

3 FREGE'S NOTE. K. Fischer, System der Logik und Metaphysik oder Wissenschaftslehre [System of Logic and Metaphysics, or Theory of Science], 2nd edn, § 94.

TRANSLATOR'S NOTE. Kuno Fischer (1824–1907), German historian of philosophy.

aticians, I think, to oppose such a view, which is apt to belittle a principal object of their science, and with it the science itself. Yet quite similar pronouncements are found among mathematicians too. On the contrary, we shall have to grant the concept of number a finer structure than most concepts of other sciences, even though it is still one of the simplest in arithmetic.

E.5 Now, to refute the delusion that there are really no difficulties at all about the positive whole numbers and that universal agreement reigns, it seemed good to me to discuss some opinions of philosophers and mathematicians on the questions at issue here. It will be seen how little concord there is to be found, so little that directly contradictory pronouncements occur. Some say, for example, "units are identical with one another"; others hold them to be different, and both have reasons for their claim that cannot be dismissed out of hand. In this way I seek to awaken the need for a more exact investigation.[4] At the same time, by first examining the views others have expressed, I want to prepare the ground for my own conception. I want the reader to be convinced in advance that those other paths do not lead to the goal, and that my opinion is not one of many with equal claims. In this way I hope to settle the question definitively, at least in its main points.

E.6 Admittedly, my discussion has thereby probably become more philosophical than many mathematicians may think proper; but a thorough investigation of the concept of number will always have to turn out somewhat philosophical. This task is common to mathematics and philosophy.

E.7 Despite many attempts from both sides, the collaboration of these sciences is not as fruitful as would be desirable and indeed possible. The reason, it seems to me, is the predominance in philosophy of psychological ways of looking at things, which have penetrated even into logic. Mathematics has no points of contact whatever with this tendency, and that easily explains the aversion of many mathematicians to philosophical

4 German *Einheit* means both "unit" (a thing counted) and "unity" (oneness); the translation chooses by sense.

reflection. When, for example, *Stricker*[5] calls the ideas of numbers motor ideas, dependent on muscular feelings, the mathematician cannot recognize his numbers in this and does not know what to make of such a proposition. An arithmetic founded on muscular feelings would certainly turn out full of feeling, but also just as blurred as this foundation. No, arithmetic has nothing whatever to do with feelings.[6] Nor has it anything to do with inner images, compounded from traces of earlier sense impressions. The fluctuating and indefinite character of all these formations stands in sharp contrast to the definiteness and fixity of mathematical concepts and objects. It may indeed be useful to study the ideas that occur in mathematical thinking and how they change; but let psychology not imagine that it can contribute anything whatever to the foundation of arithmetic. To the mathematician as such, these inner images, their origin and their alteration, are a matter of indifference. *Stricker* himself says that the word "hundred" calls up in him nothing but the sign 100. Others may picture the letter C or something else. Does it not follow that in our case these inner images are entirely indifferent and accidental to the essence of the matter, as accidental as a blackboard and a piece of chalk, and that they do not deserve to be called ideas of the number hundred at all? Let us not look for the essence of the matter in such ideas! Let us not take the description of how an idea arises for a definition, nor an account of the mental and physical conditions under which we become conscious of a proposition for a proof; and let us not confuse a proposition's being thought with its truth! It seems we must be reminded that a proposition no more ceases to be true when I stop thinking of it than the sun is annihilated when I close my eyes. Otherwise we shall end up thinking it necessary, when proving the Pythagorean theorem, to recall the phosphorus content of our brain; and an astronomer will shrink

5 FREGE'S NOTE. Studien über Association der Vorstellungen [Studies on the Association of Ideas]. Vienna 1883.

6 Salomon Stricker (1834–1898), Austrian physiologist and pathologist. "Idea" translates *Vorstellung*, which Frege always uses in the subjective, psychological sense.

from extending his conclusions to times long past, for fear of the objection: "You calculate there that 2 · 2 = 4; but the idea of number has a development, a history! One may doubt whether it had already got that far back then. How do you know that this proposition already held in that distant past? Might not the beings living then have had the proposition 2 · 2 = 5, from which the proposition 2 · 2 = 4 only developed through natural selection in the struggle for existence, a proposition that is perhaps destined in its turn to evolve by the same route into 2 · 2 = 3?" Est modus in rebus, sunt certi denique fines! [There is a measure in things; there are, in the end, fixed limits!] The historical approach, which tries to eavesdrop on the becoming of things and to recognize their essence from their becoming, is certainly fully justified; but it also has its limits. If nothing fixed and eternal persisted in the constant flux of all things, the world would cease to be knowable and everything would be plunged into confusion.[7] People seem to imagine that concepts arise in the individual mind the way leaves grow on trees, and they think they can recognize the essence of concepts by investigating their origin and trying to explain them psychologically from the nature of the human mind. But this conception draws everything into the subjective, and, followed through to the end, abolishes truth. What is called the history of concepts is probably either a history of our knowledge of concepts or a history of the meanings of words. Often it is only through great intellectual labor, which can go on for centuries, that we succeed in recognizing a concept in its purity, in peeling it out of the alien wrappings that hid it from the mind's eye. What, then, are we to say when someone, instead of continuing this work where it does not yet seem complete, treats it as worthless and goes off to the nursery, or transports himself back to the earliest conceivable stages of human development, in order to discover there, like *J. S. Mill*, some gingerbread or pebble arithmetic! All that is missing is to ascribe a special significance for the concept of number to the pleasant taste of the gingerbread. This is surely

7 Horace, *Satires* I.1.106.

the very opposite of a rational procedure, and at any rate as unmathematical as possible. No wonder mathematicians want nothing to do with it! Instead of finding a special purity of concepts where one believes oneself to be near their source, one sees everything blurred and undifferentiated, as through a fog.[8] It is as if someone wanted to get to know America by putting himself back in the position of Columbus when he caught the first doubtful glimmer of his supposed India. Such a comparison proves nothing, of course; but I hope it makes my meaning clear. It may well be that in many cases the history of discoveries is useful as a preparation for further research; but it must not try to take its place.

E.8 For mathematicians, it would hardly have been necessary to combat such views. But since I wanted to settle the disputed questions treated here as far as possible for philosophers too, I was forced to engage a little with psychology, if only to repel its incursion into mathematics.

E.9 Incidentally, psychological turns of phrase occur in mathematical textbooks too. When an author feels obliged to give a definition but cannot, he wants at least to describe the way one arrives at the object or concept in question. Such cases are easily recognized, because the explanation is never referred back to later on. For teaching purposes, leading the student toward the thing is quite in order; only it should always be clearly distinguished from a definition. E. *Schröder*[9] supplies an amusing example of how mathematicians, too, can confuse the grounds of a proof with the inner or outer conditions for carrying it out. Under the heading "Sole Axiom" he offers the following: "The principle in question might well be called the axiom of the inherence of signs.[10] It gives us the certainty that in all our developments and deductions the signs stick in our memory—but even more firmly to the paper," etc.

8 John Stuart Mill (1806–1873), British philosopher, whose *System of Logic* (1843) treats arithmetic as an empirical science.

9 FREGE'S NOTE. Lehrbuch der Arithmetik und Algebra [Textbook of Arithmetic and Algebra].

10 Ernst Schröder (1841–1902), German mathematician and logician.

E.10 But however firmly mathematics must refuse any help from psychology, it can just as little deny its close connection with logic. Indeed, I agree with those who consider a sharp separation impracticable. This much will be granted: every investigation into the cogency of a proof or the legitimacy of a definition must be logical. But such questions cannot possibly be dismissed from mathematics, since only by answering them can the necessary certainty be reached.

E.11 In this direction too, admittedly, I go somewhat beyond what is usual. In investigations of this kind most mathematicians are satisfied once they have met the immediate need. If a definition lends itself willingly to proofs, if no contradictions are encountered anywhere, if connections between apparently remote matters come to light, and if a higher order and regularity results, then the definition is usually regarded as sufficiently secured, and little is asked about its logical justification. This procedure at least has the merit that one does not easily miss the goal entirely. I too think that definitions must prove themselves by their fruitfulness, by making it possible to carry out proofs with them. But we must bear in mind that the rigor of a proof remains an illusion, however gapless the chain of inference, if the definitions are justified only after the fact, by our not having run into any contradiction. In that case we have basically attained only an empirical certainty, and must really be prepared to meet in the end with a contradiction that brings the whole edifice crashing down. That is why I believed I had to go back to the general logical foundations somewhat further than most mathematicians perhaps think necessary.

THREE FUNDAMENTAL PRINCIPLES

Frege states the three principles that guide the whole investigation.

E.12 In this investigation I have kept to the following fundamental principles:

E.13 always to separate sharply the psychological from the logical, the subjective from the objective;

E.14 never to ask for the meaning of a word in isolation, but only in the context of a sentence;

E.15 never to lose sight of the distinction between concept and object.

E.16 To comply with the first, I have always used the word "idea" in the psychological sense, and have distinguished ideas from concepts and objects. If the second principle is disregarded, one is almost forced to take inner images or acts of the individual mind as the meaning of words, and so to violate the first principle as well. As for the third point, it is a mere illusion to suppose that a concept can be made into an object without being altered. From this follows the untenability of a widespread formal theory of fractions, negative numbers, etc. How I envisage the improvement I can only indicate in this work. In all these cases, as with the positive whole numbers, what matters will be to establish the sense of an equation.

E.17 My results will, I think, meet with the agreement of those mathematicians who take the trouble to consider my reasons, at least in the main. They seem to me to be in the air, and perhaps each of them individually has already been stated, at least approximately; but in this connection with one another they may well be new. I have sometimes been surprised that accounts which come so close to my conception on one point diverge so sharply from it on others.

E.18 Among philosophers the reception will vary with their standpoint. It will probably be worst among those empiricists who recognize induction as the only original mode of inference, and not even that as a mode of inference, but as habituation. Perhaps one or another will take this occasion to subject the foundations of his theory of knowledge to a fresh examination. To those who might declare my definitions unnatural, I would point out that the question here is not whether they are natural, but whether they go to the heart of the matter and are logically unobjectionable.

E.19 I indulge the hope that philosophers too, if they examine it without prejudice, will find something useful in this work.

THE DEMAND FOR RIGOROUS PROOF

Mathematics is returning to strict proof, which leads back to the concept of Number.

1.1 After mathematics had for a time strayed from Euclidean rigor, it is now returning to it, and is even striving to go beyond it. In arithmetic, simply because many of its procedures and concepts are of Indian origin, a laxer way of thinking had become customary than in geometry, which was developed chiefly by the Greeks. The invention of higher analysis only encouraged this. On the one hand, a rigorous treatment of these doctrines met with considerable, almost insuperable difficulties; on the other, overcoming them seemed hardly to repay the effort. Yet further developments have shown ever more clearly that in mathematics a merely moral conviction, supported by many successful applications, is not enough. Proof is now demanded for many things that formerly passed as self-evident. Only in this way have the limits of validity been established in many cases. The concepts of function, continuity, limit and the infinite have turned out to need sharper determination. Negative and irrational numbers, which had long been admitted into science, have had to submit to a closer examination of their legitimacy.

1.2 Thus we see everywhere the endeavor to prove rigorously, to draw the limits of validity precisely, and, in order to be able to do so, to grasp concepts sharply.

2.1 Pursued further, this path must lead to the concept of Number and to the simplest propositions that hold of positive whole numbers, which form the foundation of the whole of arithmetic. Of course, numerical formulas such as 5 + 7 = 12, and laws such as the associative law of addition, are so abundantly confirmed by the countless applications made of them every day that it may seem almost ridiculous to call them into question by demanding a proof. But it lies in the very nature of mathematics that wherever a proof is possible, it prefers proof to confirma-

tion by induction. Euclid proves many things that anyone would grant him anyway.[11] It was by refusing to be satisfied even with Euclidean rigor that people were led to the investigations connected with the parallel axiom.

2.2 Thus this movement toward the greatest possible rigor has in many respects already gone beyond the need first felt, and that need has steadily grown in extent and intensity.

2.3 For the purpose of a proof is not only to place the truth of a proposition beyond all doubt, but also to afford insight into how truths depend on one another. Once we have convinced ourselves that a boulder is immovable, by trying in vain to move it, we may further ask what it is that supports it so securely. The further we pursue these investigations, the fewer the primitive truths to which we reduce everything; and this simplification is in itself a goal worth striving for. Perhaps, too, a further hope will be borne out: that by bringing to consciousness what people have done instinctively in the simplest cases, and extracting what is universally valid in it, we may arrive at general methods of forming concepts or of establishing grounds that can be used in more complicated cases as well.

3.1 Philosophical motives, too, have led me to such investigations. Here the questions whether arithmetical truths are a priori or a posteriori, synthetic or analytic, await their answer. For although these concepts themselves belong to philosophy, I believe the question cannot be decided without the help of mathematics. Of course, this depends on the sense one gives to these questions.

3.2 It is not uncommon to arrive first at the content of a proposition and then to give the rigorous proof by another, more laborious route, through which we often also come to know the conditions of validity more precisely. In general, therefore, we must separate the question of how we arrive at the content of a judgment from the question of where we get the justification for our assertion.

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11 "Number" with a capital N translates *Anzahl*: the number that answers the question "how many?"; lower-case "number" translates *Zahl*.

3.3 Now those distinctions between a priori and a posteriori, synthetic and analytic, concern, in my[12] view, not the content of the judgment but the justification for making the judgment. Where there is no such justification, the possibility of that classification falls away too. An a priori error is then as much an absurdity as, say, a blue concept. When one calls a proposition a posteriori or analytic in my sense, one is not judging the psychological, physiological and physical conditions that made it possible to form the content of the proposition in consciousness. Nor is one judging how someone else came, perhaps erroneously, to hold it true. One is judging what, at the deepest level, the justification for holding it true rests on.

3.4 This removes the question from the domain of psychology and assigns it to that of mathematics, where a mathematical truth is concerned. The point now is to find the proof and to follow it back to the primitive truths. If on this path one comes only upon the general laws of logic and upon definitions, one has an analytic truth. It is presupposed here that the propositions on which the admissibility of a definition may rest are also taken into account. But if the proof cannot be given without using truths that are not of a general logical nature but relate to a special domain of knowledge, then the proposition is synthetic. For a truth to be a posteriori, its proof must be unable to do without appealing to facts, that is, to unprovable truths that lack generality and contain assertions about particular objects. If, on the other hand, the proof can be given entirely from general laws which themselves neither admit of proof nor need it, then the truth is a priori.[13]

4.1 Starting from these philosophical questions, we arrive at the same demand that has grown up independently within

12 FREGE'S NOTE. I do not, of course, mean to read a new sense into them, but only to capture what earlier writers, and *Kant* in particular, meant.

mathematics itself: to prove the fundamental propositions of arithmetic with the utmost rigor, if at all possible. For only if every gap in the chain of inference is most carefully avoided can we say with certainty which primitive truths the proof rests on; and only when we know these shall we be able to answer those questions.

4.2 Now if we try to meet this demand, we very soon come to propositions that cannot be proved so long as we do not succeed in analyzing the concepts occurring in them into simpler ones or in reducing them to something more general. Here it is above all Number that must either be defined or be recognized as indefinable. That is to be the task of this book.[14] The decision about the nature of arithmetical laws will depend on how it is accomplished.

4.3 Before tackling these questions themselves, I shall say a few things in advance that may give hints toward answering them. For if there turn out to be reasons, from other points of view, for holding that the fundamental propositions of arithmetic are analytic, these reasons also speak for their provability and for the definability of the concept of Number. Reasons for the a posteriori character of these truths will have the opposite effect. So let us first give these points of dispute a preliminary examination.

13 FREGE'S NOTE. If we recognize general truths at all, we must also grant that there are such primitive laws, since nothing follows from individual facts alone, except on the basis of a law. Even induction rests on the general proposition that this procedure can establish the truth of a law, or at least a probability for it. For anyone who denies this, induction is nothing more than a psychological phenomenon, a way in which people come to believe in the truth of a proposition without that belief being in any way justified by it.

14 FREGE'S NOTE. In what follows, therefore, unless otherwise stated, no numbers will be discussed except the positive whole numbers, which answer the question "how many?"

I. Views of Certain Writers on the Nature of Arithmetical Propositions

Meinungen einiger Schriftsteller über die Natur der arithmetischen Sätze
Are numerical formulas provable? Are the laws of arithmetic inductive truths, or synthetic a priori, or analytic? Kant, Leibniz, Mill.

SIND DIE ZAHLFORMELN BEWEISBAR?

5.1 We must distinguish numerical formulas that deal with particular numbers, like 2 + 3 = 5, from the general laws that hold for all whole numbers.

5.2 Some philosophers[1] hold the former to be unprovable and immediately evident, like axioms. *Kant*[2] declares them unprovable and synthetic, but hesitates to call them axioms, because they are not general and because there are infinitely many of them. *Hankel*[3] rightly calls this assumption of infinitely many unprovable primitive truths inappropriate and paradoxical. It does indeed conflict with reason's need to have the first foundations within view. And is it really immediately evident that

1 FREGE'S NOTE. Hobbes, Locke, Newton. Cf. *Baumann*, Die Lehren von Zeit, Raum und Mathematik [The Doctrines of Time, Space and Mathematics], pp. 241 and 242, pp. 365 ff., p. 475.

TRANSLATOR'S NOTE. Johann Julius Baumann (1837–1916), German philosopher, whose *Die Lehren von Zeit, Raum und Mathematik* (1868–69) collects passages from earlier philosophers; Frege often quotes them from it.

2 FREGE'S NOTE. Kritik der reinen Vernunft [Critique of Pure Reason], ed. Hartenstein, III, p. 157.

3 FREGE'S NOTE. Vorlesungen über die complexen Zahlen und ihre Functionen [Lectures on Complex Numbers and Their Functions], p. 55.

135664 + 37863 = 173527? No! And *Kant* cites precisely this in support of the synthetic nature of these propositions. But it speaks rather against their unprovability; for how else are we to see that they are true, if not by a proof, since they are not immediately evident?[4] *Kant* wants to call on the intuition of fingers or points for help, and so runs the risk of making these propositions look empirical, contrary to his own view; for the intuition of 37863 fingers is certainly not a pure one. The term "intuition" does not seem quite apt either, since even 10 fingers can give rise to the most varied intuitions, depending on their positions relative to one another. Do we in fact have an intuition of 135664 fingers or points at all? If we had, and had one of 37863 fingers and one of 173527 fingers, then the correctness of our equation would have to be immediately evident, at least for fingers, if it were unprovable; but this is not the case.

5.3 *Kant* evidently had only small numbers in mind. Then the formulas for large numbers would be provable, while those for small numbers would be immediately evident through intuition. But it is awkward to make a fundamental distinction between small and large numbers, especially since no sharp boundary could be drawn between them. If the numerical formulas were provable from, say, 10 onwards, one would rightly ask: why not from 5 onwards, from 2, from 1?

6.1 Other philosophers and mathematicians have accordingly asserted that the numerical formulas are provable. *Leibniz*[5] says:

6.2 "It is not an immediate truth that 2 and 2 are 4, assuming that 4 designates 3 and 1. It can be proved, as follows: Definitions: 1) 2 is 1 and 1, 2) 3 is 2 and 1, 3) 4 is 3 and 1. Axiom: If identicals are substituted for one another, the equation still holds. Proof: 2 + 2 = 2 + 1 + 1 = 3 + 1 = 4. (Def. 1. Def. 2. Def. 3.) Therefore, by the axiom: 2 + 2 = 4."

6.3 At first this proof seems to be built up entirely from definitions and the axiom cited. The axiom, too, could be turned into a

4 Hermann Hankel (1839–1873), German mathematician.

definition, as *Leibniz* himself did elsewhere[6]. It seems that we need know nothing more about 1, 2, 3, 4 than what is contained in the definitions. On closer inspection, however, we discover a gap, concealed by the omission of brackets. To be more exact, we should have to write: 2 + 2 = 2 + (1 + 1) = (2 + 1) + 1 = 3 + 1 = 4.

6.4 What is missing here is the proposition 2 + (1 + 1) = (2 + 1) + 1, which is a special case of a + (b + c) = (a + b) + c. If this law is presupposed, it is easy to see that every formula of elementary addition can be proved in this way. Each number is then to be defined from the one before it. Indeed, I do not see how the number 437986, say, could be given to us more fittingly than in Leibniz's way. In this way we get it into our power, even without having an idea of it. By such definitions the infinite set of numbers is reduced to one and to increase by one, and each of the infinitely many numerical formulas can be proved from a few general propositions.

6.5 This is also the view of *H. Grassmann* and *H. Hankel*. The former wants to obtain the law a + (b + 1) = (a + b) + 1 through a definition, saying[7]:[8]

6.6 "If a and b are any members of the basic series, then by the sum a + b we understand that member of the basic series for which the formula a + (b + e) = a + b + e holds."

6.7 Here e is to mean the positive unit. Two objections can be raised against this definition. First, the sum is defined by means of itself. If we do not yet know what a + b is supposed to mean, we do not understand the expression a + (b + e) either. But this objection can perhaps be removed by saying, admittedly against the wording, that what is to be defined is not the sum but addition. It could then still be objected that a + b would be an empty

5 FREGE'S NOTE. B: Nouveaux Essais [New Essays], IV, § 10. Erdm. p. 363.
TRANSLATOR'S NOTE. "Erdm." refers to J. E. Erdmann's 1840 edition of Leibniz's philosophical works.

6 FREGE'S NOTE. C: Non inelegans specimen demonstrandi in abstractis [A not inelegant example of proof in abstract matters]. Erdm. p. 94.

7 FREGE'S NOTE. Lehrbuch der Mathematik für höhere Lehranstalten [Textbook of Mathematics for Secondary Schools], Part I: Arithmetik, Stettin 1860, p. 4.

8 Hermann Grassmann (1809–1877), German mathematician and linguist.

sign if there were no member of the basic series of the required kind, or more than one. *Grassmann* simply presupposes, without proving it, that this is not the case, so that the rigor is merely apparent.

MILL'S PEBBLE ARITHMETIC

Mill claims that the definitions of numbers assert observed physical facts; Frege disputes this.

7.1 One would think that the numerical formulas are synthetic or analytic, a posteriori or a priori, according as the general laws on which their proof rests are so. This, however, is opposed by the view of *John Stuart Mill*. At first, admittedly, he seems to want, like *Leibniz*, to base the science on definitions,[9] since he defines the individual numbers as Leibniz does. But his prejudice that all knowledge is empirical immediately spoils the correct thought again. For he informs us[10] that those definitions are not definitions in the logical sense: they do not merely fix the meaning of an expression, but at the same time assert an observed fact. What in the world can the observed or, as *Mill* also puts it, physical fact be that is asserted in the definition of the number 777864? Of all the wealth of physical facts that opens up before us here, *Mill* names only a single one, the fact supposedly asserted in the definition of the number 3. According to him, it consists in this: that there are collections of objects which, while they make this impression $^{0}_{0}{}^{0}$ on the senses, can be separated into two parts, as follows: ◦◦ ◦. How fortunate that not everything in the world is nailed down tight; otherwise we could not carry out this separation, and 2 + 1 would not be 3! What a pity that *Mill* did not also illustrate the physical facts underlying the numbers 0 and 1!

7.2 *Mill* continues: "This proposition being granted, we call all such parts 3." From this we learn that it is really incorrect to

9 FREGE'S NOTE. A System der deductiven und inductiven Logik [A System of Deductive and Inductive Logic], trans. J. Schiel, Book III, ch. XXIV, § 5.
10 FREGE'S NOTE. Op. cit., Book II, ch. VI, § 2.

speak of three strokes when the clock strikes three, or to call sweet, sour and bitter three sensations of taste. Nor is the expression "three ways of solving an equation" acceptable; for none of these ever gives us a sensory impression like that of $^{0}_{0}{}^{0}$.

7.3 Now *Mill* says: "The calculations do not follow from the definition itself, but from the observed fact." But at what point in the proof of the proposition 2 + 2 = 4 given above should *Leibniz* have appealed to that fact? *Mill* fails to point out where the gap is, although he gives a proof of the proposition 5 + 2 = 7 that corresponds exactly to Leibniz's.[11] The gap that really exists, which lies in the omission of the brackets, he overlooks, just as *Leibniz* does.

7.4 If the definition of each individual number really asserted a special physical fact, we could never sufficiently admire a man who calculates with nine-digit numbers for his knowledge of physics. Perhaps, however, *Mill*'s view is not that all these facts must be observed individually, but that it is enough to have derived by induction a general law in which they are all included. But try to state this law, and you will find it impossible. It is not enough to say: there are large collections of things that can be broken up. For this does not say that there are collections as large, and of the kind, as are required for the definition of, say, the number 1000000; nor does it specify the manner of division any more exactly. Mill's conception necessarily leads to the demand that a fact be specially observed for each number, because a general law would lose precisely what is peculiar to the number 1000000, which necessarily belongs to its definition. Indeed, according to *Mill*, we could not set 1000000 = 999999 + 1 unless we had observed precisely this peculiar way of breaking up a collection of things, which differs from the way that belongs to any other number.

8.1 *Mill* seems to think that the definitions 2 = 1 + 1, 3 = 2 + 1, 4 = 3 + 1, etc., may not be made until the facts he mentions have been observed. Indeed, we must not define 3 as (2 + 1) if we attach no sense at all to (2 + 1). The question, however, is

11 FREGE'S NOTE. Op. cit., Book III, ch. XXIV, § 5.

whether this requires observing that collection and its separation. If so, the number 0 would be a puzzle; for, so far as I know, no one has yet seen or touched 0 pebbles. *Mill* would certainly declare 0 to be something senseless, a mere figure of speech. Calculations with 0 would be a mere game with empty signs, and the only wonder would be how anything rational could come of it. But if these calculations have a serious meaning, then the sign 0 itself cannot be entirely senseless either. And this opens up the possibility that 2 + 1, like 0, might still have a sense even if the fact *Mill* mentions had not been observed. Who, indeed, will claim that the fact which, according to *Mill*, is contained in the definition of an 18-digit number has ever been observed? And who will deny that such a numeral nevertheless has a sense?

8.2 Perhaps it will be said that the physical facts are needed only for the smaller numbers, say up to 10, since the rest can be composed from these. But if we can form 11 from 10 and 1 merely by definition, without having seen the corresponding collection, there is no reason why we cannot compose 2 from 1 and 1 in the same way. If calculations with the number 11 do not follow from a fact characteristic of that number, how is it that calculations with 2 must rest on the observation of a certain collection and its peculiar separation?

8.3 It may be asked how arithmetic could exist if we could distinguish no things at all through the senses, or only three. Such a state of affairs would certainly be awkward for our knowledge of arithmetical propositions and their applications—but would it be awkward for their truth? If we call a proposition empirical because we must have made observations in order to become conscious of its content, then we are not using the word "empirical" in the sense in which it is opposed to "a priori". We are then making a psychological assertion, which concerns only the content of the proposition; whether it is true does not come into question. In that sense all of Münchhausen's stories are empir-

ical too; for one must certainly have observed all sorts of things to be able to invent them.[12]

ARE THE LAWS OF ARITHMETIC INDUCTIVE TRUTHS?

Frege argues that the general laws of arithmetic are neither laws of nature nor inductive generalizations.

9.1 The considerations so far make it probable that the numerical formulas can be derived solely from the definitions of the individual numbers by means of a few general laws, and that these definitions neither assert observed facts nor presuppose them for their legitimacy. What matters, then, is to recognize the nature of those laws.

9.2 For his proof of the formula 5 + 2 = 7, mentioned earlier, *Mill*[13] wants to use the proposition "whatever is made up of parts is made up of the parts of those parts." He regards this as a more characteristic expression of the proposition otherwise known in the form "the sums of equals are equal." He calls it an inductive truth and a law of nature of the highest order. It is typical of the imprecision of his exposition that he does not bring this proposition in at all at the point in the proof where, in his own view, it is indispensable. Still, it seems that his inductive truth is meant to stand in for *Leibniz*'s axiom: "if identicals are substituted for one another, the equation still holds." But in order to be able to call arithmetical truths laws of nature, *Mill* reads into them a sense they do not have. He thinks, for example,[14] that the equation 1 = 1 could be false, because one pound weight does not always have exactly the same weight as another. But that is not at all what the proposition 1 = 1 means to assert.

9.3 *Mill* understands the + sign as expressing the relation of the parts of a physical body, or of a heap, to the whole; but that is

12 Baron Münchhausen, the boastful teller of impossible tall tales in eighteenth-century German and English story collections, loosely based on Hieronymus von Münchhausen (1720–1797).

13 FREGE'S NOTE. Op. cit., Book III, ch. XXIV, § 5.

14 FREGE'S NOTE. Op. cit., Book II, ch. VI, § 3.

not the sense of this sign. 5 + 2 = 7 does not mean that if one pours 2 volumes of liquid into 5 volumes of liquid one obtains 7 volumes of liquid. That is an application of the proposition, and it is admissible only if no change of volume occurs, say as the result of a chemical reaction. *Mill* constantly confuses the applications that can be made of an arithmetical proposition—which are often physical and presuppose observed facts—with the purely mathematical proposition itself. In some applications the plus sign may indeed seem to correspond to the formation of a heap; but that is not its meaning. For in other applications there can be no question of heaps, aggregates, or the relation of a physical body to its parts—for example, when the calculation concerns events. True, one can speak of parts here too; but then the word is used not in the physical or geometrical sense but in the logical sense, as when one calls the assassinations of heads of state a part of murders in general. Here we have logical subordination. And so addition, in general, does not correspond to a physical relation either. Consequently the general laws of addition cannot be laws of nature either.

10.1 But perhaps they could still be inductive truths. How would that be conceived? From what facts are we supposed to start in order to rise to the general? These can surely only be the numerical formulas. In that case, of course, we would lose again the advantage we gained from the definitions of the individual numbers, and we would have to look around for some other way of grounding the numerical formulas. Even if we set aside this not altogether trivial difficulty, we still find the ground unfavorable for induction; for here we lack the uniformity that can otherwise give this procedure great reliability. *Leibniz*[15] already has Philalèthe's assertion—"The various modes of number are capable of no other difference than that of more or less; hence they are simple modes, like those of space"—answered as follows:[16]

10.2 "That can be said of time and of the straight line, but by no means of figures, and still less of numbers, which are not merely

15 FREGE'S NOTE. Baumann, op. cit., vol. II, p. 39; Erdm. p. 243.

different in magnitude but also dissimilar. An even number can be divided into two equal parts, and an odd one cannot; 3 and 6 are triangular numbers, 4 and 9 are squares, 8 is a cube, and so on. And this holds even more of numbers than of figures; for two unequal figures can be perfectly similar to each other, but two numbers never."

10.3 We have, admittedly, grown used to regarding numbers as homogeneous in many respects; but that is only because we know a great many general propositions that hold of all numbers. Here, however, we must take our stand where none of these has yet been recognized. In fact it would be hard to find an example of an inductive inference corresponding to our case. Elsewhere we are often helped by the proposition that every place in space and every point in time is, in and of itself, as good as any other. An effect must occur just as well at another place and at another time, provided only that the conditions are the same. That falls away here, because numbers are spaceless and timeless. The positions in the number series are not of equal value, as the places in space are.

10.4 Numbers also behave quite differently from, say, the individuals of an animal species. They have an order of rank determined by the nature of the case; each is formed in its own way and has its own peculiarity, which stands out especially in 0, 1 and 2. Elsewhere, when a proposition about a genus is established by induction, one usually already has a whole series of common properties from the definition of the generic concept alone. Here it is hard to find even a single one that would not itself first have to be proved.

10.5 Our case might most easily be compared with the following. Suppose that in a borehole the temperature has been found to increase regularly with depth, and that very different rock strata have been encountered so far. Then, clearly, nothing can be inferred about the nature of the deeper strata from the observations made at this borehole alone, and whether the regularity

16 Philalèthe is the spokesman for Locke's views in Leibniz's *New Essays on Human Understanding*; Théophile, who answers him, speaks for Leibniz.

of the temperature distribution would continue to hold must be left open. Both what has been observed so far and what lies deeper do indeed fall under the concept "what is encountered as drilling continues"; but that can be of little use here. Just as little will it help us, in the case of numbers, that they all fall under the concept "what is obtained by continually increasing by one". One may find a difference between the two cases in this: the strata are merely encountered, whereas the numbers are actually created by continued increase by one and determined thereby in their whole essence. This can only mean that from the way a number, e.g. 8, has arisen by increase by 1, all its properties can be derived. But in saying this one concedes, at bottom, that the properties of numbers follow from their definitions. And the possibility opens up of proving the general laws of numbers from the mode of generation common to all of them, while the particular properties of the individual numbers would be inferred from the particular way in which each is formed by continued increase by one. So too, in the case of the earth's strata, whatever is already determined by the depth alone at which they are met with—namely their relations of position—can be inferred from just that, without any need for induction; but whatever is not determined by it, induction cannot teach either.

10.6 Presumably the procedure of induction can itself be justified only by means of general propositions of arithmetic—unless by induction one means mere habit. For habit has no power whatever to guarantee truth. Scientific procedure, following objective standards, sometimes finds a high probability established by a single confirmation, and sometimes considers a thousandfold recurrence almost worthless; habit, by contrast, is determined by the number and strength of impressions and by subjective circumstances that have no right at all to influence judgment. Induction must rest on the theory of probability, since it can never make a proposition more than probable. But how this theory could be developed without presupposing arithmetical laws is impossible to see.

11.1 *Leibniz*,[17] by contrast, holds that necessary truths, such as are found in arithmetic, must have principles whose proof does not depend on examples, and therefore not on the testimony of the senses—although without the senses it would never have occurred to anyone to think of them. "The whole of arithmetic is innate in us, and is in us virtually." What he means by the expression "innate" is made clear by another passage:[18] "It is not true that everything one learns is not innate—the truths of numbers are in us, and nonetheless one learns them, whether by drawing them from their source, when one learns them by way of proof (which shows precisely that they are innate), or …".

SYNTHETIC A PRIORI OR ANALYTIC?

Kant's appeal to intuition is examined, and arithmetic is contrasted with geometry.

12.1 If we add the opposition between analytic and synthetic, we obtain four combinations, one of which, however—namely analytic a posteriori—drops out. If one has decided with *Mill* for a posteriori, there is thus no further choice; so for us only the possibilities synthetic a priori and analytic remain to be considered. *Kant* decides for the former. In that case there is presumably nothing left but to appeal to a pure intuition as the ultimate ground of knowledge, although it is hard to say here whether it is a spatial or a temporal one, or what else it might be. *Baumann*[19] agrees with *Kant*, though on somewhat different grounds. According to *Lipschitz*[20] too, the propositions asserting that the Number is independent of the manner of counting, and that the summands can be interchanged and grouped, flow from inner intuition.[21] *Hankel*[22] bases the theory of the real numbers

17 FREGE'S NOTE. Baumann, op. cit., vol. II, pp. 13 and 14; Erdm. p. 195, pp. 208 and 209.
18 FREGE'S NOTE. Baumann, op. cit., vol. II, p. 38; Erdm. p. 212.
19 FREGE'S NOTE. Op. cit., vol. II, p. 669.
20 FREGE'S NOTE. Lehrbuch der Analysis, vol. I, p. 1.
21 Rudolf Lipschitz (1832–1903), German mathematician at Bonn, author of a *Lehrbuch der Analysis* (Textbook of Analysis, 1877–80).
22 FREGE'S NOTE. Theorie der complexen Zahlensysteme, pp. 54 and 55.

on three principles, to which he ascribes the character of *notiones communes* [common notions]: "They become completely evident through explication, hold for all domains of magnitude according to the pure intuition of magnitude, and can, without losing their character, be turned into definitions by saying: by the addition of magnitudes is understood an operation that satisfies these propositions." The last assertion contains an unclarity. Perhaps one can make the definition; but it cannot serve as a substitute for those principles. For in applying it the question would always be: are Numbers magnitudes, and is what we usually call the addition of Numbers addition in the sense of this definition? And to answer that, one would already have to know those propositions about Numbers. Further, the expression "pure intuition of magnitude" is objectionable. If one considers everything that is called a magnitude—Numbers, lengths, areas, volumes, angles, curvatures, masses, velocities, forces, intensities of light, strengths of galvanic current, and so on—one can well understand how all this can be subsumed under a *concept* of magnitude; but the expression "intuition of magnitude", let alone "pure intuition of magnitude", cannot be accepted as apt. I cannot even grant an intuition of 100000, still less of number in general, let alone of magnitude in general. People appeal to inner intuition too readily when they cannot give any other ground. But in doing so one should not entirely lose sight of the sense of the word "intuition".

12.2 In the Logic (ed. Hartenstein, VIII, p. 88), *Kant* defines:

12.3 "An intuition is a singular idea (repraesentatio singularis [singular representation]), a concept a general idea (repraesentatio per notas communes [representation through common marks]) or reflected idea (repraesentatio discursiva [discursive representation])."

12.4 Here the relation to sensibility is not expressed at all—the relation that is surely thought in addition in the Transcendental Aesthetic, and without which intuition cannot serve as a principle of knowledge for synthetic a priori judgments. In the Critique of Pure Reason (ed. Hartenstein III, p. 55) we read:

12.5 "By means of sensibility, therefore, objects are given to us, and it alone supplies us with intuitions."

12.6 The sense of our word in the Logic is accordingly wider than in the Transcendental Aesthetic. In the logical sense one could perhaps call 100000 an intuition; for it is not a general concept. But taken in this sense, intuition cannot serve to ground the laws of arithmetic.

13.1 In general it will be well not to overestimate the kinship with geometry. I have already cited a passage from Leibniz against it. A geometrical point, considered by itself, cannot be distinguished in any way from any other; the same holds of straight lines and planes. Only when several points, lines or planes are grasped together in a single intuition does one distinguish them. If in geometry general propositions are obtained from intuition, this is explained by the fact that the intuited points, lines and planes are really not particular ones at all, and can therefore count as representatives of their whole kind. With numbers the matter is different: each has its own peculiarity. To what extent a particular number can represent all the others, and where its particularity makes itself felt, cannot be said offhand.

14.1 A comparison of truths with respect to the domain they govern also tells against the empirical and synthetic nature of the arithmetical laws.

14.2 Empirical propositions hold for physical or psychological actuality; geometrical truths govern the domain of what is spatially intuitable, whether it be actuality or a product of the imagination. The wildest fever-dreams, the boldest inventions of legend and of poets—which make animals talk and stars stand still, turn stones into people and people into trees, and teach how to pull yourself out of a swamp by your own hair—are still bound to the axioms of geometry, so long as they remain intuitable. Only conceptual thought can in a certain way break free of these axioms, when, say, it assumes a space of four dimensions or of positive curvature. Such considerations are by no means useless; but they leave the ground of intuition entirely. Even if

intuition is enlisted to help here, it is still always the intuition of Euclidean space, the only space of whose configurations we have any intuition. It is then simply taken not as it is but as a symbol for something else; for example, one calls straight or flat what one nonetheless intuits as curved. For conceptual thought one can always assume the opposite of this or that geometrical axiom without becoming entangled in contradictions with oneself when one draws conclusions from such assumptions, which conflict with intuition. This possibility shows that the geometrical axioms are independent of one another and of the primitive laws of logic, and are therefore synthetic. Can the same be said of the fundamental principles of the science of number? Does not everything fall into confusion if one tries to deny one of these? Would thought still be possible at all? Does not the ground of arithmetic lie deeper than that of all empirical knowledge, deeper even than that of geometry? The arithmetical truths govern the domain of the countable. This is the most comprehensive of all; for not only the actual, not only the intuitable belongs to it, but everything thinkable. Should not the laws of number, then, stand in the most intimate connection with the laws of thought?

15.1 That *Leibniz*'s pronouncements can only be read in favor of the analytic nature of the laws of number is to be expected, since for him the a priori coincides with the analytic. Thus he says[23] that algebra borrows its advantages from a far higher art, namely true logic. In another passage[24] he compares necessary and contingent truths with commensurable and incommensurable magnitudes, and holds that in the case of necessary truths a proof, or a reduction to identities, is possible. But these remarks lose weight because *Leibniz* inclines to regard all truths as provable:[25] "... that every truth has its a priori proof, drawn from the concept of the terms, although it is not always in our power to arrive at this analysis." The comparison with commensurability

23 FREGE'S NOTE. Baumann, op. cit., vol. II, p. 56; Erdm. p. 424.
24 FREGE'S NOTE. Baumann, op. cit., vol. II, p. 57; Erdm. p. 83.
25 FREGE'S NOTE. Baumann, op. cit., vol. II, p. 57; Pertz, II, p. 55.

and incommensurability, however, once again sets up a barrier between contingent and necessary truths—one that, for us at least, cannot be crossed.

15.2 *W. Stanley Jevons* speaks out very decidedly in favor of the analytic nature of the laws of number:[26] "Number is only logical discrimination, and algebra a highly developed logic."[27]

16.1 But this view too has its difficulties. Is this towering, widely branching and still growing tree of the science of number to be rooted in mere identities? And how do the empty forms of logic manage to extract such content from themselves?

16.2 *Mill* thinks: "The doctrine that by the skillful handling of language we discover facts that can reveal hidden processes of nature is so contrary to common sense that it takes some progress in philosophy before one can believe it."

16.3 Certainly so—if one thinks nothing while doing the skillful handling. *Mill* is attacking here a formalism that hardly anyone holds. Everyone who uses words or mathematical signs claims that they mean something, and no one will expect anything meaningful to come out of empty signs. But it is possible for a mathematician to carry out quite long calculations without understanding by his signs anything perceptible by the senses, anything intuitable. That does not make these signs meaningless; one still distinguishes their content from the signs themselves, even if the content can perhaps be grasped only by means of the signs. One is aware that other signs could have been laid down for the same thing. It is enough to know how the content made sensible in the signs is to be treated logically, and —if one wants to make applications to physics—how the transition to the phenomena must be made. But the real sense of the propositions is not to be seen in such an application. In it a large part of the generality is always lost, and something particular enters that is replaced by something else in other applications.

17.1 For all the disparagement of deduction, one cannot deny that the laws established by induction are not enough. From

26 FREGE'S NOTE. The principles of science. London 1879. p. 156.

them new propositions must be derived that are not contained in any one of them singly. That these propositions are already, in a certain way, contained in all of them together does not release us from the work of developing them out of the laws and setting them out on their own. This opens up the following possibility. Instead of tying a chain of inferences directly to a fact, one can leave the fact undecided and carry its content along as a condition. By thus replacing all the facts in a train of thought with conditions, one will obtain the result in a form in which an outcome is made dependent on a series of conditions. This truth would be established by thought alone—or, to put it in *Mill*'s terms, by the skillful handling of language. It is not impossible that the laws of number are of this kind. They would then be analytic judgments, even though they need not have been found by thought alone; for what matters here is not the way they were found but the kind of grounds on which their proof rests. Or, as *Leibniz* says,[28] "the question here is not the history of our discoveries, which differs in different people, but the connection and natural order of truths, which is always the same." Observation would then, in the end, have to decide whether the conditions contained in a law established in this way are fulfilled. In this way one would finally arrive at just the point one would have reached by tying the chain of inferences directly to the observed facts. But the procedure indicated here is in many cases preferable, because it leads to a general proposition that need not be applicable only to the facts at hand. The truths of arithmetic would then be related to those of logic much as the theorems of geometry are related to its axioms. Each would contain within itself a whole chain of inferences, condensed for future use, and its usefulness would consist in our no longer having to draw the inferences one by one, but being able to state the result of the

27 William Stanley Jevons (1835–1882), English economist and logician, author of *The Principles of Science* (1874).

28 FREGE'S NOTE. Nouveaux Essais, IV, § 9; Erdm. p. 360.

whole chain at once.[29] In view of the enormous development of arithmetical theory and its manifold applications, the widespread contempt for analytic judgments, and the fairy tale of the sterility of pure logic, will then, of course, no longer be tenable.

17.2 If this view—which is not voiced here for the first time—could be carried through in detail so rigorously that not the slightest doubt remained, that would not, it seems to me, be an altogether unimportant result.

29 FREGE'S NOTE. It is striking that *Mill* too, op. cit., Book II, ch. VI, § 4, seems to express this view. His good sense does break through his prejudice in favor of the empirical from time to time. But that prejudice keeps throwing everything into confusion again, by making him confuse the physical applications of arithmetic with arithmetic itself. He seems not to know that a hypothetical judgment can be true even when the condition is not true.

II. Views of Certain Writers on the Concept of Number

Meinungen einiger Schriftsteller über den Begriff der Anzahl
Is number a property of external things? Is it something subjective? Number as a set or aggregate.

18.1 Turning now to the original objects of arithmetic, we distinguish the individual numbers 3, 4, etc. from the general concept of Number. We have already decided that the individual numbers are best derived, in the manner of *Leibniz*, *Mill*, *H. Grassmann* and others, from one and increase by one, but that these definitions remain incomplete so long as one and increase by one are left undefined. We have seen that general propositions are needed in order to derive the numerical formulas from these definitions. Precisely because of their generality, such laws cannot follow from the definitions of the individual numbers, but only from the general concept of Number. We now subject this concept to closer examination. In the course of this, presumably, one and increase by one will also have to be discussed, and so the definitions of the individual numbers too may expect to be supplemented.

19.1 Here I would like at once to oppose the attempt to conceive of number geometrically, as a ratio of lengths or areas. People evidently believed they would make the many applications of arithmetic to geometry easier by placing the beginnings of the two in the closest relation from the start.

19.2 *Newton*[1] wants to understand by number not so much a set of units as the abstract ratio of any magnitude to another mag-

nitude of the same kind, taken as unit. One may grant that this aptly describes number in the wider sense, which also includes fractions and irrational numbers; but it presupposes the concepts of magnitude and of the ratio of magnitudes. It seems, then, that a definition of number in the narrower sense, of Number, would not be made superfluous. For *Euclid* needs the concept of the equimultiple in order to define the identity of two ratios of lengths; and the equimultiple in turn comes down to an identity of numbers. But it may be that the identity of ratios of lengths can be defined independently of the concept of number. But one would then remain uncertain how the number so defined geometrically stood in relation to the number of everyday life. The latter would then be completely cut off from science. And yet one may surely demand of arithmetic that it supply the points of attachment for every application of number, even if the application itself is not its business. Ordinary calculation, too, must find the grounding of its procedure in science. And then the question arises whether arithmetic itself can manage with a geometrical concept of number, when one thinks of the Number of roots of an equation, of the numbers that are prime to a given number and smaller than it, and of similar cases. By contrast, the number that gives the answer to the question "how many?" can also determine how many units a length contains. Calculation with negative, fractional and irrational numbers can be reduced to calculation with the natural numbers. Perhaps, though, *Newton* meant to understand by the magnitudes whose ratio number is defined to be not only geometrical magnitudes but also sets. In that case, however, the definition becomes useless for our purpose; for of the two expressions "number by which a set is determined" and "ratio of a set to the unit of sets", the second gives no better information than the first.

20.1 The first question, then, will be whether number is definable. *Hankel*[2] declares against it: "What it means to think or posit

1 FREGE'S NOTE. Baumann, op. cit., vol. I, p. 475.

2 FREGE'S NOTE. Theorie der complexen Zahlensysteme, p. 1.

an object once, twice, three times … cannot be defined, given the fundamental simplicity of the concept of positing." Here, however, what matters is less the positing than the once, twice, three times. If that could be defined, the indefinability of positing would trouble us little. *Leibniz* is inclined to regard number, at least approximately, as an adequate idea, that is, as one so distinct that everything occurring in it is in turn distinct.

20.2 If on the whole people tend rather to regard Number as indefinable, this is probably due more to the failure of attempts to define it than to any counter-arguments drawn from the matter itself.

IS NUMBER A PROPERTY OF EXTERNAL THINGS?

Number is compared with color and shown not to belong to things as their properties do.

21.1 Let us at least try to assign Number its place among our concepts! In language, numbers mostly appear in adjectival form and in attributive combination, much like the words "hard", "heavy", "red", which mean properties of external things. The question naturally arises whether the individual numbers must also be conceived in this way, and whether, accordingly, the concept of Number could be classed with, say, that of color.

21.2 This seems to be the view of *M. Cantor*,[3] when he calls mathematics an empirical science insofar as it takes its start from the consideration of objects of the external world. Number, on his account, arises only through abstraction from objects.[4]

21.3 *E. Schröder*[5] has number copied from actuality and taken from it, the units being represented by ones. This he calls abstracting number. In this representation, he says, the units

3 FREGE'S NOTE. Grundzüge einer Elementarmathematik, p. 2, § 4. Similarly Lipschitz, Lehrbuch der Analysis, Bonn 1877, p. 1.

4 Moritz Cantor (1829–1920), German historian of mathematics—not to be confused with Georg Cantor, the founder of set theory.

5 FREGE'S NOTE. Lehrbuch der Arithmetik und Algebra, Leipzig 1873, pp. 6, 10 and 11.

are presented only with regard to their frequency, all other determinations of the things, such as color and shape, being disregarded. Here "frequency" is just another expression for Number. Schröder thus puts frequency, or Number, on a level with color and shape and regards it as a property of things.

22.1 *Baumann*[6] rejects the thought that numbers are concepts abstracted from external things: "Because external things do not present us with strict unities; they present us with bounded groups or sensible points, but we are free to regard these in turn as many." Indeed: whereas I cannot change the color of a thing or its hardness in the slightest by a mere way of regarding it, I can regard the Iliad as one poem, as 24 books, or as a large Number of lines. Does one not speak of 1000 leaves in quite a different sense from that in which one speaks of the green leaves of the tree? We ascribe the green color to each leaf, but not so the number 1000. We can bring all the leaves of the tree together under the name of its foliage. This too is green, but it is not 1000. To what, then, does the property 1000 really belong? It almost seems to belong neither to the single leaf nor to the totality; perhaps not really to the things of the external world at all? If I give someone a stone with the words "Determine the weight of this", I have thereby given him the whole object of his investigation. But if I put a pack of playing cards into his hand with the words "Determine the Number of this", he does not know whether I want to know the number of cards, or of complete decks, or perhaps of units of value in the game of skat.[7] By putting the pack into his hand I have not yet given him the object of his investigation completely; I must add a word: "card", "deck", "unit of value". Nor can one say that the different numbers exist side by side here in the way the different colors do. I can point to a single colored surface without saying a word, but not so to a single number. If I can with equal right call an object green and red, that is a sign that this object is not the real bearer of the green. That I have only in a surface that is green and nothing

6 FREGE'S NOTE. Op. cit., vol. II, p. 669.

7 Skat is a German card game in which the cards carry point values.

else. Likewise, an object to which I can with equal right ascribe different numbers is not the real bearer of a number.

22.2 An essential difference between color and Number thus lies in this: the blue color belongs to a surface independently of our choice. It is a capacity to reflect certain rays of light and to absorb others to a greater or lesser degree, and our way of regarding it cannot change that in the slightest. By contrast, I cannot say that the Number 1, or 100, or any other belongs to the pack of playing cards in itself, but at most in relation to our arbitrary way of regarding it—and even then not in such a way that we could simply ascribe the Number to it as a predicate. What we choose to call a complete deck is obviously an arbitrary stipulation, and the pack of playing cards knows nothing about it. But by considering the pack in this respect we may discover that we can call it two complete decks. Someone who did not know what is called a complete deck would probably find some other Number in it sooner than two.

23.1 *Mill*[8] answers the question of what number belongs to as a property as follows:

23.2 "The name of a number denotes a property belonging to the aggregate of things that we call by the name; and this property is the characteristic way in which the aggregate is made up of parts, or can be separated into parts."

23.3 Here, to begin with, the definite article in the expression "the characteristic way" is a mistake; for there are very different ways in which an aggregate can be divided, and one cannot say that one alone is characteristic. A bundle of straw, for example, can be divided by cutting through all the straws, or by breaking it up into single straws, or by making two bundles out of it. Is a heap of a hundred grains of sand made up in the same way as a bundle of 100 straws? And yet we have the same number. The number word "one" in the expression "one straw" surely does not express how this straw is made up of cells or of molecules. The number 0 causes still more difficulty. Must the straws form a bundle at all in order to be counted? Must all the blind people

8 FREGE'S NOTE. Op. cit., Book III, ch. XXIV, § 5.

in the German Empire be gathered together in one assembly before the expression "number of blind people in the German Empire" can have a sense? Are a thousand grains of wheat, once they have been sown, no longer a thousand grains of wheat? Are there really such things as aggregates of proofs of a theorem, or of events? And yet these too can be counted. And it makes no difference whether the events are simultaneous or thousands of years apart.

24.1 This brings us to another reason for not classing number with color and solidity: its far greater applicability.

24.2 *Mill*[9] thinks that the truth that whatever is made up of parts is made up of parts of those parts holds of all natural phenomena, since all of them can be counted. But cannot far more than that be counted? *Locke*[10] says: "Number applies to human beings, angels, actions, thoughts, every thing that exists or can be imagined." *Leibniz*[11] rejects the opinion of the Scholastics that number cannot be applied to incorporeal things, and calls number a kind of incorporeal figure, arising from the union of any things whatever—for example God, an angel, a human being, and motion, which together are four. Hence, he holds, number is something entirely universal and belongs to metaphysics. In another passage[12] he says: "What does not have force and power cannot be weighed; what has no parts accordingly has no measure; but there is nothing that does not admit of number. Thus number is, as it were, the metaphysical figure."

24.3 It would indeed be a marvel if a property abstracted from external things could be transferred to events, to ideas, to concepts, without any change of sense. It would be just as if one wanted to speak of a fusible event, a blue idea, a salty concept, a viscous judgment.

24.4 It is absurd that what is by its nature sensible should occur in what is not sensible. When we see a blue surface, we have a

9 FREGE'S NOTE. Op. cit., Book III, ch. XXIV, § 5.
10 FREGE'S NOTE. Baumann, op. cit., vol. I, p. 409.
11 FREGE'S NOTE. Ibid., vol. II, p. 56.
12 FREGE'S NOTE. Ibid., vol. II, p. 2.

distinctive impression that corresponds to the word "blue"; and we recognize it again when we catch sight of another blue surface. If we wanted to assume that, in the same way, something sensible corresponded to the word "three" when we look at a triangle, then we would have to find this again in three concepts too; something non-sensible would have something sensible about it. One may well grant that a kind of sense impression corresponds to the word "triangular"; but then one must take this word as a whole. We do not see the three in it directly; rather, we see something to which a mental activity can attach, leading to a judgment in which the number 3 occurs. With what, for instance, do we perceive the Number of syllogistic figures that Aristotle sets out? With our eyes, perhaps? At most we see certain signs for these syllogistic figures, not the figures themselves. How are we supposed to see their Number if they themselves remain invisible? But, it may be said, it is enough to see the signs; their number is identical with the number of syllogistic figures. How, then, does one know that? For this one must surely already have determined the latter in some other way. Or is the proposition "the Number of syllogistic figures is four" merely another way of saying "the Number of signs for the syllogistic figures is four"? No! Nothing is meant to be said about the signs; nobody wants to know anything about the signs unless their property at the same time expresses a property of what they signify. Since the same thing can, without any logical error, have different signs, the number of signs need not even agree with the number of things signified.

25.1 Whereas for *Mill* number is something physical, for *Locke* and *Leibniz* it exists only in idea. It is true, as *Mill*[13] says, that two apples are physically different from three apples, and two horses from one horse—a different visible and tangible phenomenon.[14] But are we to infer from this that twoness, three-

13 FREGE'S NOTE. Op. cit., Book III, ch. XXIV, § 5.

14 FREGE'S NOTE. Strictly speaking one would have to add: insofar as they are a phenomenon at all. But if someone has one horse in Germany and one in America (and no others), he owns two horses. These, however, do not form a phenomenon; only each horse by itself could be called one.

ness, is something physical? A pair of boots can be the same visible and tangible phenomenon as *two* boots. Here we have a difference in number to which no physical difference corresponds; for *two* and *one pair* are by no means the same thing, as *Mill* oddly seems to believe. And finally, how is it possible for two concepts to differ physically from three concepts?

25.2 Thus *Berkeley*[15] says: "It is to be noted that number is nothing fixed and settled, existing realiter [really] in things themselves. It is entirely a creature of the mind, when it considers either an idea by itself or a combination of ideas to which it wants to give a name, and so lets it count as a unit. According as the mind combines its ideas in varying ways, the unit varies; and as the unit varies, so too does number, which is only a collection of units. A window = 1; a house in which there are many windows = 1; many houses make up one city."[16]

IS NUMBER SOMETHING SUBJECTIVE?

Frege argues that number is objective, like the North Sea or the earth's axis, and not an idea.

26.1 Following this line of thought, one easily comes to regard number as something subjective. The way in which number arises in us seems capable of shedding light on its nature. Everything would then depend on a psychological investigation. It is presumably in this sense that *Lipschitz*[17] says:

26.2 "Whoever wants to gain an overview of certain things will begin with one particular thing and keep adding a new thing to the earlier ones." This seems to fit much better the way we obtain, say, the intuition of a constellation than the formation of numbers. The aim of gaining an overview is inessential; for one can hardly say that a herd becomes easier to survey once one learns how many head it consists of.

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15 FREGE'S NOTE. Baumann, op. cit., vol. II, p. 428.

16 George Berkeley (1685–1753), Irish philosopher; the passage comes from his *Essay toward a New Theory of Vision* (1709).

17 FREGE'S NOTE. Lehrbuch der Analysis, p. 1. I assume that *Lipschitz* has an inner process in mind.

26.3 Such a description of the inner processes that precede the making of a judgment of number can never, even if it is more accurate, replace a genuine definition of the concept. It can never be drawn on in the proof of an arithmetical proposition; it teaches us no property of numbers. For number is no more an object of psychology, or a product of mental processes, than, say, the North Sea is. It does nothing to diminish the objectivity of the North Sea that it depends on our choice which part of the earth's general covering of water we mark off and decide to call "North Sea". That is no reason for wanting to investigate this sea by psychological means. So number too is something objective. If one says "the North Sea is 10,000 square miles in area", then neither by "North Sea" nor by "10,000" does one point to a state or process within oneself; rather, one asserts something entirely objective, which is independent of our ideas and the like. If at some other time we chose to draw the boundaries of the North Sea somewhat differently, or to understand something else by "10,000", it would not be the same content, previously correct, that now became false; rather, a false content would perhaps have been substituted for a true one—and the truth of the former would in no way be canceled by that.

26.4 When the botanist states the Number of petals of a flower, he means to say something just as factual as when he states its color. The one depends on our whim as little as the other. So there is a certain similarity between Number and color. But it does not consist in both being perceptible by the senses in external things; it consists in both being objective.

26.5 I distinguish the objective from the palpable, the spatial, the actual. The earth's axis and the center of mass of the solar system are objective, but I would not call them actual, as the earth itself is. The equator is often called a *thought* line; but it would be wrong to call it a *thought-up* line. It did not come into being through thinking, as the result of a mental process; it is only recognized, grasped, through thinking. If being recognized were a coming into being, we could assert nothing positive

about the equator with respect to any time preceding this alleged coming into being.

26.6 According to Kant, space belongs to appearance. It would be possible for it to present itself to other rational beings quite differently from the way it presents itself to us. Indeed, we cannot even know whether it appears to one person as it does to another, for we cannot lay one person's intuition of space beside another's in order to compare them. And yet there is something objective in it. Everyone recognizes the same geometrical axioms, if only in practice, and must do so in order to find their way about in the world. What is objective in it is what is law-governed, conceptual, judgeable, what can be expressed in words. The purely intuitive is not communicable. To make this clearer, let us suppose two rational beings to whom only projective properties and relations are intuitable: three points lying on one straight line, four points lying in one plane, and so on. Suppose that what one of them intuits as a point appears to the other as a plane, and conversely. What is the line joining points for the one is the line of intersection of planes for the other, and so on, always in dual correspondence. Then they could understand each other perfectly well and would never become aware of the difference in their intuitions, because in projective geometry every theorem has a dual counterpart; a divergence in aesthetic appreciation would be no reliable sign. They would be in complete agreement about all geometrical theorems; they would merely translate the words into their intuitions differently. With the word "point", say, one would connect this intuition and the other that one. So one can still say that this word means something objective to them; only we must not understand by this meaning what is peculiar to their intuition. And in this sense the earth's axis, too, is objective.

26.7 With "white" we usually think of a certain sensation, which is of course wholly subjective. But even in ordinary usage, it seems to me, an objective sense often comes to the fore. When we call snow white, we mean to express an objective quality, which in ordinary daylight we recognize by a certain sensa-

tion. If the snow is lit with colored light, we take that into account in our judgment. We may say: it *looks* red now, but it is white. Even someone who is color-blind can speak of red and green, although he does not distinguish these colors in sensation. He recognizes the difference by the fact that others make it, or perhaps by a physical experiment. So a color word often signifies not our subjective sensation—which we cannot know to agree with anyone else's, for using the same name plainly guarantees nothing of the kind—but an objective quality. Thus by objectivity I understand independence of our sensing, intuiting and forming ideas, of our sketching inner pictures out of memories of earlier sensations; but not independence of reason. For to answer the question what things are independently of reason would be to judge without judging, to wash the fur without getting it wet.[18]

27.1 That is also why I cannot agree with *Schlömilch*[19], who calls number the idea of the position of an object in a series[20]. If number were an idea, arithmetic would be psychology. It is no more that than astronomy is. Just as astronomy is concerned not with ideas of the planets but with the planets themselves, so the object of arithmetic is not an idea either. If two were an idea, it would in the first instance be mine alone. Another person's idea

18 An allusion to the German proverb "Wash my fur, but don't make me wet", said of someone who wants a thing without its unavoidable consequences.

19 FREGE'S NOTE. Handbuch der algebraischen Analysis, p. 1.

20 FREGE'S NOTE. One can also object that the same idea of a position would then always have to appear whenever the same number occurs, which is plainly false. What follows would not apply if by "idea" he meant an objective idea; but then what difference would there be between the idea of the position and the position itself? The idea in the subjective sense is what the psychological laws of association are concerned with; it is sensory and pictorial in nature. The idea in the objective sense belongs to logic and is essentially non-sensory, although the word that means an objective idea often carries a subjective one along with it, which, however, is not its meaning. The subjective idea is often demonstrably different in different people; the objective idea is the same for all. Objective ideas can be divided into objects and concepts. To avoid confusion, I shall use "idea" only in the subjective sense. By combining both meanings in this word, Kant gave his doctrine a very subjective, idealist coloring and made it harder to hit on what he really meant. The distinction drawn here is as justified as that between psychology and logic. If only these were always kept strictly apart!

is, simply as such, a different one. We would then have perhaps many millions of twos. We would have to say: my two, your two, a two, all twos. If one admits latent or unconscious ideas, we would also have unconscious twos, which would later become conscious again.[21] With every new generation growing up, new twos would keep arising, and who knows whether over thousands of years they might not change so much that 2 × 2 = 5. Even so, it would be doubtful whether there are, as is usually supposed, infinitely many numbers. Perhaps 10^10 would be only an empty sign, and there would be no idea at all, in any being whatever, that could be called by that name.

27.2 We see what oddities result when the thought that number is an idea is spun out a little further. And we come to the conclusion that number is neither spatial and physical, like Mill's heaps of pebbles and gingerbread nuts, nor subjective like ideas, but non-sensory and objective. The ground of objectivity cannot, after all, lie in the sense impression, which, as an affection of our mind, is wholly subjective. As far as I can see, it can lie only in reason.

27.3 It would be strange if the most exact of all sciences had to rely on psychology, which is still feeling its way far too uncertainly.

NUMBER AS A SET

Definitions of Number as a set or plurality are found too vague and to exclude 0 and 1.

28.1 Some writers define Number as a set, a plurality or a multiplicity. One drawback here is that the numbers 0 and 1 are excluded from the concept. These expressions are quite indefinite. Sometimes they come closer to the meaning of "heap", "group", "aggregate", where spatial togetherness is in mind; sometimes they are used almost synonymously with "Number", only more vaguely. So no analysis of the concept of Number can

21 Oskar Schlömilch (1823–1901), German mathematician.

be found in such a definition. *Thomae*[22] requires, for the formation of number, that different names be given to different sets of objects. This obviously means a sharper determination of those sets of objects, for which the giving of names is only the outward sign. What kind of determination this is: that is the question.[23] Plainly the idea of number would not arise if we introduced names for "3 stars", "3 fingers", "7 stars" in which no common components could be discerned. What matters is not that names are given at all, but that what is numerical about them is designated on its own. For this, it must be recognized in its particularity.

28.2 The following difference must also be noted. Some call number a set of things or objects; others, as *Euclid*[24] did long ago, define it as a set of units. This expression needs special discussion.

22 FREGE'S NOTE. Elementare Theorie der analytischen Functionen, p. 1.

23 Carl Johannes Thomae (1840–1921), German mathematician and Frege's colleague at Jena.

24 FREGE'S NOTE. Beginning of Book 7 of the Elements: Μονάς έστι, καθ' ήν έκαστον τών όντων έν λέγεται. Άριθμός δέ τό έκ μονάδων συγκείμενον πλήθος. [A unit is that in virtue of which each of the things that exist is called one. And a number is the multitude composed of units.]

III. Views on Unity and One

Meinungen über Einheit und Eins
Does "one" express a property of objects? Are units identical with one another? Attempts to overcome the difficulty, and its solution.

DRÜCKT DAS ZAHLWORT »EIN« EINE EIGENSCHAFT VON GEGENSTÄNDEN AUS?

29.1 In the definitions Euclid gives at the beginning of Book 7 of the Elements, he seems to use the word "μονάς" [unit] to designate now an object to be counted, now a property of such an object, now the number one. The translation "unit" works everywhere, but only because this word itself shimmers with these different meanings.

29.2 *Schröder*[1] says: "Each of the things to be counted is called a unit." The question is why we first bring things under the concept of unit instead of simply defining: number is a set of things. But that would throw us back on the previous point. One might at first try to find a closer determination in calling things units, by following the linguistic form, treating "one" as an adjective and taking "one city" in the same way as "wise man". A unit would then be an object to which the property "one" belonged, and it would stand to "one" much as "a wise man" stands to the adjective "wise". To the reasons urged above against number's being a property of things, some special ones are added here. It would be striking, first of all, that every thing would have this property. It would be incomprehensible why

1 FREGE'S NOTE. Op. cit., p. 5.

anyone should still explicitly ascribe the property to a thing. Only the possibility that something is not wise gives the assertion that Solon is wise a sense. The content of a concept diminishes as its extension increases; if its extension becomes all-embracing, its content must be lost altogether. It is not easy to see how language would come to create an adjective that could not serve at all to determine an object more closely.[2]

29.3 If "one man" were to be understood like "wise man", we would expect that "one" could also be used as a predicate, so that, just as we say "Solon was wise", we could also say "Solon was one" or "Solon was a one". Now even if the latter expression can occur, it is not intelligible on its own. It can mean, for example, "Solon was a wise one", if "wise" is to be supplied from the context. But on its own "one" seems unable to be a predicate[3]. This shows even more clearly in the plural. While "Solon was wise" and "Thales was wise" can be combined into "Solon and Thales were wise", we cannot say "Solon and Thales were one".[4] Why this is impossible could not be understood if "one", like "wise", were a property both of Solon and of Thales.

30.1 Connected with this is the fact that no one has been able to give a definition of the property "one". When *Leibniz*[5] says, "One is what we grasp together by one act of the understanding", he is defining "one" in terms of itself. And can we not also grasp many things together by one act of the understanding? *Leibniz* concedes this in the same passage. Similarly *Baumann*[6] says: "One is what we take as one", and further: "What we posit as a point, or no longer wish to posit as divided, we regard as one; but every one of outer intuition, pure as well as empirical, we can also

2 Solon (c. 640–c. 560 BC), Athenian statesman and lawgiver, counted among the Seven Sages of Greece.

3 FREGE'S NOTE. There are turns of phrase that seem to contradict this. But on closer inspection one will find that a concept word must be supplied, or that "one" is not being used as a number word: what is meant to be asserted is not uniqueness but unity.

4 Thales of Miletus (c. 624–c. 546 BC), Greek philosopher, also counted among the Seven Sages.

5 FREGE'S NOTE. Baumann, op. cit., vol. II, p. 2; Erdm. p. 8.

6 FREGE'S NOTE. Op. cit., vol. II, p. 669.

regard as many. Every idea is one when it is marked off from another idea; but within itself it can again be distinguished into many." Thus every objective boundary of the concept is blurred, and everything depends on our way of viewing. We ask again: what sense can it have to ascribe the property "one" to any object, if, depending on our way of viewing, every object can both be one and not be one? How can a science that seeks its glory precisely in the greatest definiteness and precision rest on so hazy a concept?

31.1 Although *Baumann*[7] makes the concept of one rest on inner intuition, in the passage just quoted he nevertheless names undividedness and boundedness as its characteristic marks. If these were correct, we should expect animals, too, to be able to have some idea of unity. Does a dog, looking at the moon, have any idea, however indefinite, of what we designate by the word "one"? Hardly! And yet it certainly distinguishes individual objects. Another dog, its master, a stone it plays with, surely appear to it just as bounded, existing by themselves and undivided as they do to us. To be sure, it will notice a difference between having to defend itself against many dogs and against only one; but this is the difference that *Mill* calls physical. What would matter in particular is whether it has any awareness, however dim, of the common element we express by the word "one", say in the cases where it is bitten by one larger dog and where it chases one cat. I find that unlikely. I conclude from this that the idea of unity is not, as *Locke*[8] thinks, conveyed to the understanding by every object outside and every idea within, but is recognized by us through the higher mental powers that distinguish us from animals. In that case such properties of things as undividedness and boundedness, which animals notice just as well as we do, cannot be what is essential to our concept.

32.1 Still, we may suspect a certain connection. Language points to one by deriving "einig" [united] from "Ein" [one].

7 FREGE'S NOTE. Op. cit., vol. II, p. 669.

8 FREGE'S NOTE. Baumann, op. cit., vol. I, p. 409.

Something is the better suited to being taken as a distinct object, the more the differences within it recede compared with its differences from its surroundings, the more its inner coherence outweighs its coherence with its surroundings. So "einig" means a property that prompts us, in our way of viewing, to separate something off from its surroundings and consider it on its own. That the French "uni" means "level", "smooth" is to be explained in this way. The word "unity", too, is used in a similar way when one speaks of the political unity of a country or the unity of a work of art[9]. But in this sense "unity" belongs less with "one" than with "einig" or "einheitlich" [unified]. For when we say that the earth has one moon, we do not mean to declare it a bounded, self-standing, undivided moon. We say it in contrast to what is the case with Venus, Mars or Jupiter. In respect of boundedness and undividedness the moons of Jupiter could well bear comparison with ours, and in that sense they are just as unified.

33.1 Some writers heighten undividedness into indivisibility. *G. Köpp*[10] calls every thing, whether perceptible by the senses or not, that is thought of as indecomposable and as existing by itself an individual, and he calls the individuals to be counted "ones", where "one" is plainly used in the sense of "unit". When *Baumann* bases his view that external things do not present strict units on the ground that we are free to regard them as many, he too makes indecomposability a characteristic mark of the strict unit. By heightening inner coherence to the absolute, one obviously wants to obtain a characteristic mark of the unit that is independent of our arbitrary way of viewing. This attempt founders on the fact that almost nothing would then be left that could be called a unit and counted.[11] That is why the retreat is sounded at once: the characteristic mark is taken to be not indecomposability itself but being thought of as indecom-

9 FREGE'S NOTE. On the history of the word "Einheit", cf. Eucken, Geschichte der philosophischen Terminologie, pp. 122–3, p. 136, p. 220.

TRANSLATOR'S NOTE. Rudolf Eucken (1846–1926), German philosopher at Jena; Nobel Prize in Literature 1908.

10 FREGE'S NOTE. Schularithmetik. Eisenach 1867. pp. 5 and 6.

11 Gustav Köpp, author of a school arithmetic textbook (Eisenach, 1867).

posable. And with that we are back at the wavering way of viewing. And is anything gained by thinking of things otherwise than as they are? On the contrary! From a false assumption false conclusions may follow. But if one does not want to infer anything from indecomposability, what use is it? If one can relax the strictness of the concept a little without harm—indeed, must do so—what is this strictness for? But perhaps one is merely not supposed to think of decomposability. As if anything could be achieved by lack of thought! But there are cases where we cannot avoid thinking of decomposability at all, where an inference even rests on the unit's being composite. For example, the problem: one day has 24 hours; how many hours do 3 days have?

ARE UNITS IDENTICAL WITH ONE ANOTHER?

The units to be counted must be both identical and distinguishable, which seems contradictory.

34.1 So every attempt to define the property "one" fails, and we must presumably give up seeing a closer determination in calling things units. We return to our question: why do we call things units, if "unit" is only another name for thing, if all things are units or can be taken as such? *E. Schröder*[12] gives as the reason the identity ascribed to the objects of counting. First, it is not clear why the words "thing" and "object" could not indicate this just as well. Then the question arises: why is identity ascribed to the objects of counting? Is it merely ascribed to them, or are they really identical? In any case, no two objects are ever completely identical. On the other hand, we can almost always find some respect in which two objects agree. So we are back at the arbitrary way of viewing, unless, against the truth, we want to ascribe to things a more far-reaching identity than belongs to them. In fact many writers call units identical without qualification. *Hobbes*[13] says: "Number, said absolutely, presupposes in mathematics units identical among themselves,

12 FREGE'S NOTE. Op. cit., p. 5.
13 FREGE'S NOTE. Baumann, op. cit., vol. I, p. 242.

out of which it is produced." *Hume*[14] holds the component parts of quantity and number to be entirely homogeneous. *Thomae*[15] calls an individual of the set a unit and says: "The units are identical with one another." One could just as well, or rather more correctly, say: the individuals of the set are different from one another. What, then, is this alleged identity supposed to mean for number? The properties by which things differ are something indifferent and alien to their Number. That is why one wants to keep them out. But this does not succeed in this way. If, as *Thomae* demands, we "abstract from the peculiarities of the individuals of a set of objects", or "in considering separate things disregard the characteristic marks by which the things differ", what remains is not, as *Lipschitz* thinks, "the concept of the Number of the things considered". Rather, we obtain a general concept under which those things fall. The things themselves lose none of their particularities thereby. If, for example, in considering a white cat and a black cat I disregard the properties by which they differ, I obtain, say, the concept "cat". Even if I now bring both under this concept and call them, say, units, the white one still remains white and the black one black. Nor do the cats become colorless because I do not think of their colors, or resolve to draw no conclusions from their difference; they remain just as different as they were. The concept "cat", obtained by abstraction, does indeed no longer contain the particularities; but for that very reason it is only one.

35.1 Purely conceptual procedures do not succeed in making different things identical; and if they did succeed, we would no longer have things but only one thing. For, as *Descartes*[16] says, number—better: multiplicity—in things arises from their being distinguished. *E. Schröder*[17] rightly maintains: "The demand to count things can reasonably be made only where objects are present that appear clearly distinguishable from one another,

14 FREGE'S NOTE. Ibid., vol. II, p. 568.
15 FREGE'S NOTE. Op. cit., p. 1.
16 FREGE'S NOTE. Baumann, op. cit., vol. I, p. 103.
17 FREGE'S NOTE. Op. cit., p. 3.

for example separated in space and time and marked off from one another." Indeed, too great a similarity, as with the bars of a grating, sometimes makes counting harder. *W. Stanley Jevons*[18] expresses himself in this sense with particular sharpness: "Number is only another name for diversity. Exact identity is unity, and with diversity arises multiplicity." And further (p. 157): "It has often been said that units are units insofar as they are perfectly alike; but although they may be perfectly alike in some respects, they must differ in at least one point, otherwise the concept of multiplicity would be inapplicable to them. If three coins were so alike that they occupied the same space at the same time, they would not be three coins but one coin."

36.1 But it soon turns out that the view that units are different runs into new difficulties. *Jevons* declares: "A unit (unit) is any object of thought that can be distinguished from any other object treated as a unit in the same problem." Here unit is defined in terms of itself, and the addition "that can be distinguished from any other object" contains no closer determination, because it goes without saying. We call the object another one precisely because we can distinguish it from the first. *Jevons*[19] says further: "When I write the symbol 5, I really mean 1 + 1 + 1 + 1 + 1, and it is perfectly clear that each of these units is different from every other.[20] If required, I can designate them thus: $1' + 1'' + 1''' + 1'''' + 1'''''$." Certainly it is required to designate them differently if they are different; otherwise the greatest confusion would result. If the mere difference in the place where the one appeared were supposed to signify a difference, this would have to be laid down as a rule without exception, since otherwise we would never know whether 1 + 1 was supposed to mean 2 or 1. Then we would have to reject the equation 1 = 1 and would be in the predicament of never being able to designate the same thing a second time. That obviously will not

18 FREGE'S NOTE. The Principles of Science, 3rd ed., p. 156.
19 FREGE'S NOTE. Op. cit., p. 162.
20 Frege adds Jevons's English word "unit" in brackets to fix the sense of the German Einheit.

do. But if we want to give different things different signs, it is impossible to see why we should still keep a common component in them, rather than writing a + b + c + d + e instead of $1' + 1'' + 1''' + 1'''' + 1'''''$.

36.2 Identity, after all, is now lost, and hinting at a certain similarity is of no use. So the one melts away in our hands; we are left with the objects and all their particularities. These signs $1'$, $1''$, $1'''$ are an eloquent expression of the predicament: we need identity—hence the 1; we need difference—hence the indices, which unfortunately cancel the identity again.

37.1 We meet the same difficulty in other writers. *Locke*[21] says: "By repeating the idea of a unit and adding it to another unit, we accordingly make a collective idea, designated by the word 'two'. And whoever can do this and go on in this way, always adding one more to the last collective idea he had of a number, and can give it a name, can count." *Leibniz*[22] defines number as 1 and 1 and 1, or as units. *Hesse*[23] says: "If one can form an idea of the unit that is expressed in algebra by the sign 1, … one can also think of a second unit with equal rights, and further ones of the same kind.[24] The union of the second with the first into a whole gives the number 2."

37.2 Here we must attend to the relation in which the meanings of the words "unit" and "one" stand to each other. *Leibniz* understands by unit a concept under which one and one and one fall, as indeed he also says: "The abstract of one is unity." *Locke* and *Hesse* seem to use unit and one synonymously. At bottom *Leibniz* probably does so too; for in calling the individual objects that fall under the concept of unit all one, he designates by this word not the individual object but the concept under which they fall.

38.1 To keep confusion from setting in, however, it will be well to maintain a strict distinction between unit and one. We say "the number one", and with the definite article we indicate a

21 FREGE'S NOTE. Baumann, op. cit., vol. I, pp. 409–411.
22 FREGE'S NOTE. Baumann, op. cit., vol. II, p. 3.
23 FREGE'S NOTE. Vier Species, p. 2.
24 Ludwig Otto Hesse (1811–1874), German mathematician; his *Die vier Species* appeared in 1872.

definite, single object of scientific inquiry. There are not different numbers one, but only one. In 1 we have a proper name, which as such is as incapable of a plural as "Frederick the Great" or "the chemical element gold". It is neither chance nor imprecise notation that 1 is written without distinguishing strokes. The equation 3 − 2 = 1 would be rendered by Stanley Jevons something like this: (1′ + 1′′ + 1′′′) − (1′′ + 1′′′) = 1′. But what would be the result of (1′ + 1′′ + 1′′′) − (1′′′′ + 1′′′′′)? Certainly not 1′. It follows that on his view there would be not only different ones but also different twos, and so on; for 1′′ + 1′′′ could not be replaced by 1′′′′ + 1′′′′′. This shows very clearly that number is not an accumulation of things. Arithmetic would be abolished if, instead of the one, which is always the same, we tried to introduce different things, however similar the signs for them; after all, they could not be identical without error. Surely we cannot suppose that the deepest need of arithmetic is a faulty notation. It is therefore impossible to regard 1 as a sign for different objects, such as Iceland, Aldebaran, Solon and the like. The absurdity becomes most palpable if we think of the case where an equation has three roots, namely 2 and 5 and 4. If, following *Jevons*, we now write 1′ + 1′′ + 1′′′ for 3, then 1′ would here mean 2, 1′′ 5 and 1′′′ 4, if by 1′, 1′′, 1′′′ we understand units and hence, according to *Jevons*, the objects of thought present here. Would it not then be more intelligible to write 2 + 5 + 4 instead of 1′ + 1′′ + 1′′′?

38.2 A plural is possible only for concept words. So if we speak of "units", we cannot use this word synonymously with the proper name "one", but only as a concept word. If "unit" means "object to be counted", we cannot define number as units. If by "unit" we understand a concept under which one, and only one, falls, a plural makes no sense, and it is again impossible to define number, with *Leibniz*, as units or as 1 and 1 and 1. If we use "and" as in "Bunsen and Kirchhoff", then 1 and 1 and 1 is not 3 but 1, just as gold and gold and gold is never anything but

gold.[25] The plus sign in 1 + 1 + 1 = 3 must therefore be understood differently from the "and" that helps to designate a collection, a "collective idea".

39.1 We are thus faced with the following difficulty:

39.2 If we try to make number arise by combining different objects, we get an accumulation in which the objects are contained with precisely the properties by which they differ, and that is not number. If, on the other hand, we try to form number by combining what is identical, it keeps flowing together into one, and we never arrive at a multiplicity.

39.3 If we designate each of the objects to be counted by 1, that is a mistake, because different things receive the same sign. If we furnish the 1 with distinguishing strokes, it becomes useless for arithmetic.

39.4 The word "unit" is admirably suited to veiling this difficulty; and that is the reason—if an unconscious one—why it is preferred to the words "object" and "thing". First the things to be counted are called units, and difference gets its due. Then combining, collecting, uniting, adding, or whatever one wants to call it, passes over into the concept of arithmetical addition, and the concept word "unit" turns imperceptibly into the proper name "one". And with that we have identity. If I add an *n* to the letter *u*, and a *d* to that, anyone can easily see that this is not the number 3. But if I bring *u*, *n* and *d* under the concept "unit" and now say, instead of "*u* and *n* and *d*", "a unit and a unit and another unit" or "1 and 1 and 1", it is easy to believe that this gives us 3. The difficulty is so well hidden by the word "unit" that surely few people have any inkling of it.

39.5 Here *Mill* could rightly complain of a skillful handling of language; for here language is not the outward appearance of a thought process but merely feigns one. Here we really do get the impression that a certain mysterious power is being ascribed to words empty of thought, when different things are supposed to become identical merely by being called units.

25 Robert Bunsen (1811–1899) and Gustav Kirchhoff (1824–1887), German scientists who together developed spectral analysis.

ATTEMPTS TO OVERCOME THE DIFFICULTY

Appeals to space, time, series and abstraction fail to reconcile identity with difference.

40.1 We now consider some lines of argument that present themselves as attempts to overcome this difficulty, even if they were probably not always put forward with this aim clearly in mind.

40.2 We can first call on a property of space and time for help. A point of space, namely, cannot be distinguished at all from another, nor a straight line or plane from another, nor congruent bodies, pieces of surface or line segments from one another, when each is considered by itself. They can be distinguished only in their coexistence as components of a total intuition. So identity seems here to be united with distinguishability. The same holds of time. That is presumably why *Hobbes*[26] thinks it hardly conceivable that the identity of units should arise otherwise than by division of the continuum. *Thomae*[27] says: "If one pictures a set of individuals or units in space and counts them successively, which requires time, then, for all abstraction, there still remain as distinguishing marks of the units their different positions in space and their different succession in time."

40.3 The first objection to such a way of viewing is that the countable would then be confined to the spatial and temporal. *Leibniz*[28] already rejects the opinion of the Scholastics that number arises from the mere division of the continuum and cannot be applied to incorporeal things. *Baumann*[29] stresses the independence of number and time: the concept of unit, he says, is thinkable even without time. *Stanley Jevons*[30] says: "Three coins are three coins whether we count them one after another or regard them all at once. In many cases neither time nor space is the ground of difference, but quality alone. We can conceive of

26 FREGE'S NOTE. Baumann, op. cit., vol. I, p. 242.
27 FREGE'S NOTE. Elementare Theorie der analyt. Functionen, p. 1.
28 FREGE'S NOTE. Baumann, op. cit., vol. II, p. 2.
29 FREGE'S NOTE. Op. cit., vol. II, p. 668.
30 FREGE'S NOTE. The Principles of Science, p. 157.

the weight, inertia and hardness of gold as three properties, although none of them is before or after another, either in space or in time. Any means of distinction can be a source of plurality." I add: if the objects counted do not actually follow one another but are only counted one after another, time cannot be the ground of distinction. For in order to be able to count them one after another, we must already have distinguishing marks. Time is only a psychological requirement for counting; it has nothing to do with the concept of number. If we let non-spatial and non-temporal objects be represented by points of space or time, this may perhaps be advantageous for carrying out the count; but in principle it presupposes that the concept of number is applicable to what is non-spatial and non-temporal.

41.1 But is the aim of uniting distinguishability and identity really achieved if we disregard all distinguishing marks except the spatial and temporal ones? No! We have not come one step closer to the solution. The greater or lesser similarity of the objects makes no difference if in the end they must still be kept apart. I may no more designate the individual points, lines and so on all by 1 here than I may call them all A in geometrical arguments; for here as there it is necessary to distinguish them. Only by themselves, without regard to their spatial relations, are points of space identical with one another. But if I am to combine them, I must consider them in their spatial coexistence; otherwise they merge irretrievably into one. Taken all together, points may perhaps represent some constellation-like figure or be arranged somehow on a straight line; equal segments may perhaps meet at their endpoints to form a single segment, or lie apart from one another. The configurations arising in this way can be quite different for the same number. So here too we would have different fives, sixes and so on. Points of time are separated by short or long, equal or unequal intervals. All these are relations that have nothing whatever to do with number as such. Everywhere something particular gets mixed in, something far beneath number in its generality. Even a single moment has something peculiar to it, by which it differs, say,

from a point of space, and of which there is nothing in the concept of number.

42.1 Nor does the way out of replacing spatial and temporal arrangement by a more general concept of series lead to the goal. For position in the series cannot be the ground for distinguishing the objects, since they must already be distinguished by something in order to be arranged in a series. Such an arrangement always presupposes relations between the objects —whether spatial, temporal, logical, intervals of pitch or any others—by which we are led from one to the next, and which are necessarily bound up with their being distinguished.

42.2 When *Hankel*[31] has an object thought or posited 1 time, 2 times, 3 times, this too seems to be an attempt to unite the distinguishability of what is to be counted with its identity. But we also see at once that it is not a successful one; for these ideas or intuitions of the same object must differ in some way if they are not to flow together into one. I also think we are entitled to speak of 45 million Germans without first having thought or posited a standard German 45 million times; that might be somewhat laborious.

43.1 Probably in order to avoid the difficulties that arise if, with *Stanley Jevons*, we let each sign 1 mean one of the objects counted, E. *Schröder* wants it merely to depict an object. The consequence is that he defines only the numeral, not the number. For he says[32]: "Now, to obtain a sign capable of expressing *how many* of those units[33] are present, one directs one's attention in turn once to each of them and depicts it by a stroke: 1 (a *one*, a *single one*). These ones are set side by side in a row, but are joined to one another by the sign + (plus), since otherwise, for example, 111 would be read in the usual numerical notation as one hundred and eleven. In this way one obtains a sign such as 1 + 1 + 1 + 1 + 1, whose composition can be described by saying: '*A natural number is a sum of ones.*'"

31 FREGE'S NOTE. Theorie der complexen Zahlensysteme, p. 1.
32 FREGE'S NOTE. Lehrbuch der Arithmetik und Algebra, pp. 5 ff.
33 FREGE'S NOTE. objects to be counted.

43.2 From this we see that for Schröder number is a *sign*. What is expressed by this sign, what I have so far called number, he presupposes as known, with the words "how many of those units are present". By the word "one", too, he understands the sign 1, not its meaning. The sign + serves him at first only as an external means of connection without content of its own; addition is defined only later. He could have put it more briefly thus: one writes as many signs 1 side by side as one has objects to be counted, and connects them by the sign +. Zero would be expressed by writing nothing down.

44.1 In order not to take the distinguishing marks of things into number along with it, *Stanley Jevons*[34] says:

44.2 "It will now be not very difficult to form a clear idea of numerical abstraction. It consists in abstracting from the character of the difference from which plurality arises, retaining merely its existence. When I speak of *three men*, I need not at once specify one by one the marks by which each of them can be distinguished from each of the others. These marks must exist if they really are three men and not one and the same, and in speaking of them as many I thereby at the same time assert the existence of the requisite differences. Abstract number is thus *the empty form of difference*."

44.3 How is this to be understood? We can either abstract from the distinguishing properties of the things before combining them into a whole, or we can first form a whole and then abstract from the nature of the differences. On the first path we would never even get as far as distinguishing the things, and so we could not retain the presence of the differences either. It is the second path that *Jevons* seems to mean. But I do not believe that we would obtain the number 10000 in this way, because we are not able to apprehend so many differences at once and retain their presence; and if it happened one after another, the number would never be finished. We do count in time; but in doing so we do not obtain the number, we only determine it. Besides, stating the manner of abstracting is not a definition.

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34 FREGE'S NOTE. Op. cit., p. 158.

44.4 What are we supposed to think of under the "empty form of difference"? A proposition such as "a is different from b", perhaps, where a and b remain indeterminate? Would this proposition then be the number 2? Does the proposition "the earth has two poles" mean the same as "the North Pole is different from the South Pole"? Obviously not. The second proposition could hold without the first, and the first without the second. For the number 1000 we would then have $(1000 \cdot 999)/(1 \cdot 2)$ such propositions expressing a difference.

44.5 What *Jevons* says does not fit 0 and 1 at all, in particular. What exactly are we supposed to abstract from in order to get from the moon, for example, to the number 1? By abstracting we do obtain the concepts: satellite of the earth, satellite of a planet, heavenly body without light of its own, heavenly body, body, object. But 1 is nowhere to be met with in this series, for it is not a concept under which the moon could fall. In the case of 0 we do not even have an object from which the abstraction could start. Let no one object that 0 and 1 are not numbers in the same sense as 2 and 3! Number answers the question "how many?", and if we ask, for example, "how many moons does this planet have?" we may be prepared for the answer 0 or 1 just as well as for 2 or 3, without the sense of the question changing. The number 0 does have something special about it, to be sure, and so does 1; but at bottom that holds of every whole number, only it is less and less noticeable in the larger ones. It is entirely arbitrary to make a difference in kind here. What does not fit 0 or 1 cannot be essential to the concept of number.

44.6 Finally, assuming that number arises in this way does nothing at all to remove the difficulty we ran into when considering the notation $1' + 1'' + 1''' + 1'''' + 1''''' + 1''''''$ for 5. This notation agrees well with what *Jevons* says about the abstraction that forms number: the strokes above indicate that there is a difference, without stating what kind it is. But, as we have seen, the mere existence of difference is already enough, on *Jevons*'s view, to produce different ones, twos and threes—which is utterly incompatible with arithmetic as it stands.

SOLUTION OF THE DIFFICULTY

Frege summarizes the results so far and begins to set out his own answer.

45.1 Let us now survey what we have established so far, and the questions that have remained unanswered!

45.2 Number is not abstracted from things in the way that color, weight and hardness are; it is not a property of things in the sense that these are. The question remained: of what is something asserted by a statement of number?

45.3 Number is nothing physical, but nothing subjective either; it is not an idea.

45.4 Number does not arise by adding thing to thing. Nor does giving a name after each addition change anything here.

45.5 The expressions "plurality", "set" and "multiplicity" are unsuitable, because of their indeterminacy, for defining number.

45.6 With regard to one and unit, there remained the question of how to restrict the arbitrariness in our way of regarding things, which seemed to blur every difference between one and many.

45.7 Being delimited, being undivided and being indivisible are not usable characteristic marks for what we express by the word "one".

45.8 If we call the things to be counted units, then the unconditional assertion that the units are identical is false. That they are identical in a certain respect is correct, but worthless. Difference between the things to be counted is in fact necessary if the number is to be greater than 1.

45.9 So it seemed that we had to ascribe two contradictory properties to the units: identity and distinguishability.

45.10 A distinction must be drawn between one and unit. The word "one", as the proper name of an object of mathematical inquiry, does not admit of a plural. It is therefore senseless to

make numbers arise by putting ones together. The plus sign in 1 + 1 = 2 cannot mean such a putting together.

A STATEMENT OF NUMBER IS ABOUT A CONCEPT

Frege argues that a statement of number contains an assertion about a concept.

46.1 To shed light on the matter, it will be good to consider number in the context of a judgment, where its original manner of application comes out. If, faced with the same external phenomenon, I can say with equal truth "this is a group of trees" and "these are five trees", or "here are four companies" and "here are 500 men", then what changes is neither the individual nor the whole, the aggregate, but my naming. But that is only the sign that one concept has been replaced by another. This suggests to us, as the answer to the first question of the previous section, that a statement of number contains an assertion about a concept. This is perhaps clearest in the case of the number 0. If I say "Venus has 0 moons", then there is no moon or aggregate of moons at all of which anything could be asserted; but a property is thereby ascribed to the *concept* "moon of Venus", namely that of including nothing under it. If I say "the Emperor's carriage is drawn by four horses", then I ascribe the number four to the concept "horse that draws the Emperor's carriage".

46.2 It may be objected that a concept such as "citizen of the German Empire", although its characteristic marks remain unchanged, would have a property that changed from year to year, if the statement of number asserted such a property of it. One can reply that objects too change their properties, and this does not prevent us from recognizing them as the same. But here the reason can be given more precisely. The concept "citizen of the German Empire" contains time as a variable component; or, to put it mathematically, it is a function of time. For "a is a citizen of the German Empire" one can say "a belongs to the German Empire", and this refers to the present moment. So there is already something fluid in the concept itself. By con-

trast, the same number belongs for all eternity to the concept "citizen of the German Empire at the beginning of the year 1883, Berlin time".

47.1 That a statement of number expresses something factual, independent of our way of regarding things, can surprise only someone who takes the concept to be something subjective, like an idea. But this view is false. If, for example, we subordinate the concept of body to that of what has weight, or the concept of whale to that of mammal, we thereby assert something objective. Now if concepts were subjective, then the subordination of one to another, as a relation between them, would also be something subjective, like a relation between ideas. Admittedly, at first glance the proposition "all whales are mammals" seems to be about animals, not about concepts; but if we ask which animal is being spoken of, we cannot point to a single one. Suppose a whale is lying in front of us: our proposition still asserts nothing about it. We could not infer from it that the animal in front of us is a mammal without adding the proposition that it is a whale, and of that our proposition contains nothing. In general, it is impossible to speak of an object without designating or naming it in some way. But the word "whale" names no individual creature. If someone replies that, admittedly, what is spoken of is not a single, determinate object, but an indeterminate one, then I hold that "indeterminate object" is only another expression for "concept"—and a bad, self-contradictory one at that. Granted that our proposition can be justified only by observing individual animals, this proves nothing about its content. For the question of what it is about, it makes no difference whether it is true or not, or on what grounds we take it to be true. Now if the concept is something objective, then an assertion about it can also contain something factual.

48.1 The appearance that arose earlier with some examples, that different numbers belonged to the same thing, is explained by the fact that objects were there taken to be the bearers of number. As soon as we restore the true bearer, the concept, to its

rights, numbers prove to be just as mutually exclusive as colors are in their domain.

48.2 We now also see how people come to want to obtain number by abstraction from things. What one obtains in this way is the concept, in which one then discovers the number. Thus abstraction does in fact often precede the forming of a judgment of number. The confusion is the same as if one wanted to say: the concept of fire hazard is obtained by building a half-timbered house with a board gable and a thatched roof, whose chimneys leak.

48.3 The gathering power of the concept far surpasses the unifying power of synthetic apperception. By means of the latter it would not be possible to combine the citizens of the German Empire into a whole; but we can bring them under the concept "citizen of the German Empire" and count them.[35]

48.4 Now the wide applicability of number also becomes explicable. It is indeed puzzling how the same thing could be asserted of outer and at the same time of inner phenomena, of what is spatial and temporal and of what is neither spatial nor temporal. And in fact this does not happen at all in a statement of number. Numbers are ascribed only to the concepts under which the outer and the inner, the spatial and the temporal, the non-spatial and the non-temporal have been brought.

49.1 We find confirmation of our view in *Spinoza*, who says[36]: "I answer that a thing is called one or unique only with respect to its existence, not its essence; for we conceive things under numbers only after they have been brought to a common measure. Whoever, for example,[37] holds a sesterce and an imperial in his hand will not think of the number two unless he can give this sesterce and this imperial one and the same name, namely piece of money or coin. Then he can affirm that he has two pieces of money or coins, because he designates not only the sesterce but

35 "Synthetic apperception" is *Kant*'s term for the activity of consciousness that unites a manifold of representations into one.

36 FREGE'S NOTE. Baumann, op. cit., vol. I, p. 169.

37 Baruch Spinoza (1632–1677), Dutch philosopher; the passage comes from his letter of 1674 to Jarig Jelles (Letter 50).

also the imperial by the name coin." When he goes on: "From this it is clear that a thing is called one or unique only after another thing has been conceived that (as has been said) agrees with it", and when he holds that God cannot properly be called one or unique because we could form no abstract concept of his essence, he is mistaken in thinking that a concept can only be obtained directly, by abstraction from several objects.[38] On the contrary, one can also arrive at the concept starting from the characteristic marks; and then it is possible that no thing falls under it. If this never happened, we could never deny existence, and then the affirmation of existence would lose its content too.

50.1 E. *Schröder*[39] stresses that, if we are to be able to speak of the frequency of a thing, the name of this thing must always be a *generic name*, a general concept word (notio communis [common notion]): "For as soon as one takes an object fully into view—with all its properties and relations—it will stand alone in the world and have no equal anywhere. The name of the object will then bear the character of a *proper name* (nomen proprium [proper name]), and the object cannot be thought of as occurring repeatedly. But this holds not only of *concrete* objects; it holds generally of every thing, even if the idea of it comes about through *abstractions*, provided only that this idea contains elements sufficient to make the thing in question *completely* determinate. … The latter" (becoming an object of counting) "becomes possible for a thing only insofar as one disregards, or *abstracts* from, some of the characteristic marks and relations peculiar to it, by which it is distinguished from all other things; only then does the name of the thing become a concept applicable to several things."

51.1 What is true in this account is dressed up in such skewed and misleading expressions that it needs to be disentangled and sifted. First, it is inappropriate to call a general concept word the name of a thing. This creates the appearance that number is

38 The sesterce was an ancient Roman coin; the "imperial" (Latin *imperialis*) was a coin of the Holy Roman Empire, the imperial thaler.

39 FREGE'S NOTE. Op. cit., p. 6.

a property of a thing. A general concept word designates precisely a concept. Only with the definite article or a demonstrative pronoun does it count as the proper name of a thing, and with that it ceases to count as a concept word. The name of a thing is a proper name. An object does not occur repeatedly; rather, several objects fall under one concept. That a concept is not obtained only by abstraction from the things that fall under it has already been remarked in answer to *Spinoza*. Here I add that a concept does not cease to be a concept merely because only a single thing falls under it, a thing which is accordingly completely determined by it. To such a concept (e.g. satellite of the earth) there belongs precisely the number 1, which is a number in the same sense as 2 and 3. With a concept the question is always whether anything falls under it, and if so, what. With a proper name such questions are senseless. We must not let ourselves be deceived by the fact that language uses a proper name, e.g. moon, as a concept word, and vice versa; the distinction remains all the same. As soon as a word is used with the indefinite article, or in the plural without an article, it is a concept word.

[52.1] A further confirmation of the view that number is ascribed to concepts can be found in the German usage of saying *zehn Mann, vier Mark, drei Fass* [literally "ten man, four mark, three barrel"]. The singular here may indicate that it is the concept that is meant, not the thing. The advantage of this way of speaking comes out especially with the number 0. Otherwise, admittedly, language ascribes number to objects, not to the concept: we say "number of the bales" just as we say "weight of the bales". So we seem to be speaking of objects, when in truth we mean to assert something of a concept. This usage is confusing. The expression "four thoroughbred horses" creates the appearance that "four" qualifies the concept "thoroughbred horse" in the same way that "thoroughbred" qualifies the concept "horse". But only "thoroughbred" is a characteristic mark of this kind; by the word "four" we assert something of a concept.

53.1 By properties asserted of a concept I naturally do not mean the characteristic marks that make up the concept. These are properties of the things that fall under the concept, not of the concept. Thus "right-angled" is not a property of the concept "right-angled triangle"; but the proposition that there is no right-angled, rectilinear, equilateral triangle does state a property of the concept "right-angled, rectilinear, equilateral triangle": the number zero is ascribed to it.

53.2 In this respect existence resembles number. Affirmation of existence is, after all, nothing other than denial of the number zero. Because existence is a property of the concept, the ontological proof of the existence of God fails to reach its goal. But uniqueness is no more a characteristic mark of the concept "God" than existence is. Uniqueness cannot be used in defining this concept, just as one cannot use the solidity, roominess and comfort of a house in building it, along with the stones, mortar and beams. However, one may not conclude in general, from the fact that something is a property of a concept, that it cannot be inferred from the concept, i.e. from its characteristic marks.[40] Under certain circumstances this is possible, just as one can sometimes draw a conclusion about the durability of a building from the kind of stones it is built with. It would therefore be claiming too much to say that one can never infer uniqueness or existence from the characteristic marks of a concept; it is only that this can never happen as directly as when one ascribes a characteristic mark of a concept, as a property, to an object falling under it.

53.3 It would also be wrong to deny that existence and uniqueness could ever be characteristic marks of concepts. They are just not characteristic marks of *those* concepts to which, following language, one would like to attribute them. If, for example, we gather all concepts under which only one object falls under a single concept, then uniqueness is a characteristic mark of this concept. Under it would fall, for example, the concept "moon of

40 The ontological proof, associated with Anselm and Descartes, argues that God's existence follows from the very concept of God.

the earth", but not the so-called heavenly body itself. In this way we can let a concept fall under a higher concept—a concept of second order, so to speak. But this relation is not to be confused with that of subordination.

54.1 Now it will be possible to give a satisfactory account of the unit. *E. Schröder* says on p. 7 of the textbook mentioned: "That generic name or concept is called the denomination of the number formed in the way indicated, and constitutes the essence of its unit."

54.2 Indeed, would it not be most fitting to call a concept a unit with respect to the Number that belongs to it? We can then make sense of the assertions about the unit, that it is separated from its surroundings and indivisible. For the concept to which number is ascribed in general delimits what falls under it in a definite way. The concept "letter of the word Zahl" delimits the Z from the a, the a from the h, and so on. The concept "syllable of the word Zahl" picks out the word as a whole, and as indivisible in the sense that its parts no longer fall under the concept "syllable of the word Zahl". Not all concepts are like this. We can, for example,[41] divide what falls under the concept of red in all sorts of ways without the parts ceasing to fall under it. No finite number belongs to such a concept. The proposition that the unit is delimited and indivisible can accordingly be stated as follows:

54.3 Only a concept that delimits what falls under it in a definite way, and does not permit any arbitrary division of it, can be a unit with respect to a finite Number.

54.4 We can see, though, that indivisibility has a special meaning here.

54.5 Now we can easily answer the question of how the identity of the units is to be reconciled with their distinguishability. The word "unit" is used here in a double sense. The units are identical in the meaning of the word explained above. In the proposition "Jupiter has four moons" the unit is "moon of Jupiter". Under this concept falls I, and also II, and also III, and

41 *Zahl* is German for "number"; the example turns on its four letters and its single syllable.

also IV. Hence we can say: the unit to which I is related is identical with the unit to which II is related, and so on. There we have identity. But when we assert the distinguishability of the units, we mean by this the distinguishability of the things counted.

IV. The Concept of Number

Der Begriff der Anzahl
Every individual number is a self-subsistent object; the sense of a numerical identity; the definition of number by one-to-one correlation; zero, one and the successor; infinite numbers.

JEDE EINZELNE ZAHL IST EIN SELBSTÄNDIGER GEGENSTAND.

55.1 Now that we have recognized that a statement of number contains an assertion about a concept, we can try to complete the Leibnizian definitions of the individual numbers with definitions of 0 and 1.

55.2 It is tempting to define: the number 0 belongs to a concept if no object falls under it. But here the equivalent "no" seems to have taken the place of 0; so the following wording is preferable: the number 0 belongs to a concept if, in general, whatever a may be, the proposition holds that a does not fall under this concept.

55.3 In a similar way we could say: the number 1 belongs to a concept F if it is not the case that, in general, whatever a may be, the proposition holds that a does not fall under F; and if from the propositions "a falls under F" and "b falls under F" it follows in general that a and b are the same.

55.4 It remains to define, in general, the step from one number to the next. We try the following wording: the number (n + 1) belongs to the concept F if there is an object a that falls under F and is such that the number n belongs to the concept "falling under F, but not a".

56.1 These definitions offer themselves so naturally, given our results so far, that it needs explaining why they cannot satisfy us.

56.2 The last definition is the most likely to raise doubts. For, strictly speaking, the sense of the expression "the number n belongs to the concept G" is just as unknown to us as that of the expression "the number (n + 1) belongs to the concept F". True, by means of this definition and the one before it we can say what "the number 1 + 1 belongs to the concept F" means, and then, using that, state the sense of the expression "the number 1 + 1 + 1 belongs to the concept F", and so on. But we can never—to take a glaring example—decide by means of our definitions whether the number *Julius Caesar* belongs to a concept, or whether that well-known conqueror of Gaul is a number or not. Further, with the help of our attempted definitions we cannot prove that a = b must hold if the number a belongs to the concept F and the number b also belongs to it. The expression "*the* number that belongs to the concept F" would therefore not be justified, and this would make it impossible to prove any identity of numbers at all, since we could not grasp a determinate number in the first place. It is only an appearance that we have defined 0 and 1. In truth we have only fixed the sense of the phrases "the number 0 belongs to" and "the number 1 belongs to"; but this does not allow us to pick out 0 and 1 here as self-subsistent objects that can be recognized again.

NUMBERS AS OBJECTS

Number words are treated as proper names, and numbers as objects that cannot be pictured.

57.1 This is the place to look somewhat more closely at our way of putting it, that a statement of number contains an assertion about a concept. In the proposition "the number 0 belongs to the concept F", 0 is only part of the predicate, if we regard the concept F as the real subject. That is why I have avoided calling a number such as 0, 1 or 2 a *property* of a concept. It is precisely

because the individual number forms only part of the assertion that it appears as a self-subsistent object. I have already drawn attention above to the fact that we say "the 1" (German *die 1*), and with the definite article present 1 as an object. This self-subsistence shows itself everywhere in arithmetic, e.g. in the equation 1 + 1 = 2. Since what matters to us here is to grasp the concept of number in the form in which it is usable for science, we must not be put off by the fact that in everyday usage number also appears as an attribute. This can always be avoided. For example, the proposition "Jupiter has four moons" can be recast as "the number of Jupiter's moons is four". Here the "is" must not be regarded as a mere copula, as in the proposition "the sky is blue". This is shown by the fact that we can say "the number of Jupiter's moons is the four" or "is the number 4". Here "is" has the sense of "is identical with", "is the same as". So we have an equation asserting that the expression "the number of Jupiter's moons" designates the same object as the word "four". And the form of the equation is the dominant one in arithmetic. It is no objection to this view that the word "four" contains nothing about Jupiter or about moons. Nor is there anything about discovering or about America in the name "Columbus", and yet the same man is called Columbus and the discoverer of America.

58.1 It might be objected that we can form no idea[1] whatever of the object we call four, or the Number of Jupiter's moons, as something self-subsistent. But it is not the self-subsistence we have given to number that is to blame for this. True, it is easy to believe that in the idea of the four pips on a die there is something corresponding to the word "four"; but that is an illusion. Think of a green meadow, and see whether the idea changes if you replace the indefinite article by the number word "one". Nothing is added, whereas something in the idea does correspond to the word "green". If you picture the printed word "gold", you will not at first think of any number. If you now ask yourself how many letters it consists of, the number 4 results; but the idea does not thereby become any more determinate—it can

1 FREGE'S NOTE. "Idea" (*Vorstellung*) taken in the sense of something image-like.

remain completely unchanged. The added concept "letter of the word gold" is precisely that in which we discover the number. With the four pips on a die the matter is somewhat more hidden, because the similarity of the pips forces the concept on us so immediately that we hardly notice it stepping in. Number can be pictured neither as a self-subsistent object nor as a property in an external thing, because it is neither something perceptible by the senses nor a property of an external thing. The matter is perhaps clearest with the number 0. You will try in vain to picture 0 visible stars. You can, to be sure, think of the sky as completely covered with clouds; but there is nothing in this that corresponds to the word "star" or to 0. You are only picturing a situation that can prompt the judgment: no star can be seen now.

59.1 Every word perhaps calls up some idea in us, even a word like "only"; but the idea need not correspond to the content of the word, and in other people it may be quite different. One will then perhaps picture a situation that invites a sentence in which the word occurs; or the spoken word may call the written word to mind.

59.2 This happens not only with particles. There is surely no doubt that we have no idea of our distance from the sun. For even if we know the rule for how many times we must lay off a measuring rod, every attempt to sketch, by this rule, a picture that comes even roughly close to what is intended fails. But that is no reason to doubt the correctness of the calculation by which the distance was found, and it in no way prevents us from basing further inferences on the fact of this distance.

60.1 Even so concrete a thing as the earth we cannot picture as we have come to know it to be; instead we make do with a ball of moderate size that serves us as a sign for the earth, though we know that this is very different from it. Now although our idea often entirely misses what is intended, we still judge with great certainty about an object such as the earth, even where size comes into question.

60.2 Thinking very often carries us beyond what can be pictured, without our thereby losing the basis for our inferences.

Even if, as it seems, thinking without ideas is impossible for us humans, the connection of these ideas with what is thought can still be entirely external, arbitrary and conventional.

60.3 So the fact that the content of a word cannot be pictured is no reason to deny it any meaning or to exclude it from use. The appearance to the contrary probably arises because we consider words in isolation and ask for their meaning, which we then take to be an idea. A word for which we lack a corresponding inner picture thus seems to have no content. But we must always keep a complete sentence in view. Only in a sentence do words really have a meaning. The inner pictures that may float before us need not correspond to the logical components of the judgment. It is enough if the sentence as a whole has a sense; it is through this that its parts, too, receive their content.

60.4 This remark seems to me fitted to throw light on many difficult concepts, such as that of the infinitesimal[2], and its scope is probably not limited to mathematics.

60.5 The self-subsistence I claim for number is not meant to imply that a number word designates something outside the context of a sentence; I mean by it only to exclude its use as a predicate or attribute, which somewhat alters its meaning.

61.1 But, it may be objected, even if the earth cannot really be pictured, it is still an external thing with a definite place; but where is the number 4? It is neither outside us nor inside us. Understood in a spatial sense, that is correct. To assign a location to the number 4 makes no sense; but all that follows is that it is not a spatial object, not that it is no object at all. Not every object is somewhere. Our ideas[3], too, are not inside us in this sense (under the skin). There are ganglion cells, blood corpuscles and the like in there, but no ideas. Spatial predicates do not apply to them: one idea is neither to the right nor to the left of another; ideas are not at distances from one another that

2 FREGE'S NOTE. What matters is to define the sense of an equation such as df(x) = g(x) dx, not to exhibit a segment bounded by two distinct points whose length would be dx.

3 FREGE'S NOTE. The word understood purely psychologically, not psychophysically.

could be stated in millimeters. If we nonetheless say they are inside us, what we mean is that they are subjective.

61.2 But even if what is subjective has no place, how is it possible for the objective number 4 to be nowhere? Well, I maintain that there is no contradiction in this whatever. It is in fact exactly the same for everyone who concerns himself with it; but that has nothing to do with being in space. Not every objective object has a place.

THE SENSE OF A NUMERICAL IDENTITY

Words mean something only in the context of a sentence; Number is to be fixed through identity statements.

62.1 How, then, is a number to be given to us, if we cannot have any idea or intuition of it? Only in the context of a sentence do words mean something. So it will be a matter of defining the sense of a sentence in which a number word occurs. At first this still leaves a great deal to arbitrary choice. But we have already established that number words are to be understood as standing for self-subsistent objects. This gives us a class of sentences that must have a sense: sentences that express a recognition. If the sign a is to designate an object for us, we must have a criterion that decides in every case whether b is the same as a, even if it is not always in our power to apply this criterion. In our case we must define the sense of the sentence "the number that belongs to the concept F is the same as the number that belongs to the concept G"; that is, we must render the content of this sentence in another way, without using the expression "the Number that belongs to the concept F". In doing so we give a general criterion for the identity of numbers. Once we have acquired in this way a means of grasping a determinate number and recognizing it as the same, we can give it a number word as its proper name.

63.1 *Hume*[4] already names such a means: "When two numbers are so combined that the one always has a unit corresponding to every unit of the other, we declare them equal." In recent times

4 FREGE'S NOTE. Baumann, op. cit., vol. II, p. 565.

the view seems to have found wide favor among mathematicians[5] that the identity of numbers must be defined by means of one-to-one correlation. But logical doubts and difficulties arise first, and we must not pass them by without examination.[6]

63.2 The relation of identity does not occur only with numbers. From this it seems to follow that it must not be defined specially for this case. One would think that the concept of identity was already settled beforehand, and that from it, together with the concept of Number, it would then have to follow when Numbers are identical with one another, without any further special definition being needed.

63.3 Against this it must be pointed out that for us the concept of Number is not yet settled, but is first to be determined by means of our definition. Our aim is to construct the content of a judgment that can be taken as an equation in such a way that each side of the equation is a number. So we do not mean to define identity specially for this case, but to use the already familiar concept of identity to obtain what is to be regarded as identical. This does seem, admittedly, a very unusual kind of definition, one that logicians have probably not yet given enough attention to; but a few examples may show that it is not unheard of.

64.1 The judgment "the line a is parallel to the line b", in symbols a ∥ b, can be taken as an equation. If we do this, we obtain the concept of direction and say: "the direction of the line a is identical with the direction of the line b". So we replace the sign ∥ by the more general =, distributing the particular content of the former between a and b. We split up the content in a way different from the original one, and thereby obtain a new concept.

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5 FREGE'S NOTE. Cf. E. Schröder, op. cit., pp. 7 and 8; E. Kossak, *Die Elemente der Arithmetik* [The Elements of Arithmetic], program of the Friedrichs-Werder Gymnasium, Berlin, 1872, p. 16; G. Cantor, *Grundlagen einer allgemeinen Mannichfaltigkeitslehre* [Foundations of a General Theory of Manifolds], Leipzig, 1883.

TRANSLATOR'S NOTE. Georg Cantor (1845–1918), mathematician and founder of set theory.

6 David Hume (1711–1776); the passage is from *A Treatise of Human Nature*, Book I, Part III, Section 1.

Often, admittedly, the matter is taken the other way round, and many teachers define: parallel lines are lines with the same direction. The proposition "if two lines are parallel to a third, they are parallel to each other" can then be proved very conveniently by appeal to the similarly worded proposition about identity. Only it is a pity that this turns the true state of affairs on its head! For everything geometrical must surely be originally intuitive. Now I ask whether anyone has an intuition of the direction of a line. Of the line, certainly! But in the intuition of this line, does one also distinguish its direction? Hardly! This concept is found only through a mental activity that takes intuition as its starting point. On the other hand, we do have an idea of parallel lines. That proof works only by sleight of hand, presupposing what is to be proved through the use of the word "direction"; for if the proposition "if two lines are parallel to a third, they are parallel to each other" were incorrect, a ∥ b could not be turned into an equation.

64.2 In the same way we can obtain from the parallelism of planes a concept corresponding to that of direction for lines. I have seen the name "orientation" used for it. From geometrical similarity there arises the concept of shape, so that, for example, instead of "the two triangles are similar" we say "the two triangles have the same shape" or "the shape of the one triangle is identical with the shape of the other". In the same way we can also obtain, from the collinear relationship between geometrical figures, a concept for which there is probably as yet no name.

65.1 Now, to get from parallelism[7], for example, to the concept of direction, we try the following definition:

65.2 the proposition "the line a is parallel to the line b" is to mean the same as "the direction of the line a is identical with the direction of the line b".

65.3 This definition departs from the usual in that it seems to determine the already familiar relation of identity, while in

truth it is meant to introduce the expression "the direction of the line a", which occurs in it only incidentally. From this springs a second doubt: whether such a stipulation might not entangle us in contradictions with the familiar laws of identity. What are these laws? They can be developed as analytic truths from the concept itself. Now *Leibniz* defines[8]: "Eadem sunt, quorum unum potest substitui alteri salva veritate" [Those things are the same of which one can be substituted for the other without loss of truth]. I adopt this definition for identity. Whether one says "the same", like *Leibniz*, or "identical", is immaterial. "The same" does seem to express complete agreement, and "identical" only agreement in this or that respect; but we can adopt a way of speaking in which this difference disappears—for example, by saying, instead of "the segments are identical in length", "the length of the segments is identical" or "the same", and instead of "the surfaces are identical in color", "the color of the surfaces is identical". And that is how we used the word in the examples above. Now universal substitutability does in fact contain all the laws of identity.

65.4 To justify our attempted definition of the direction of a line, we would therefore have to show that the direction of a can everywhere be replaced by the direction of b, if the line a is parallel to the line b. This is simplified by the fact that, to begin with, we know of no assertion about the direction of a line other than its agreement with the direction of another line. So we would only need to prove substitutability in such an identity, or in contents that would contain such identities as components[9]. All other assertions about directions would first have to be defined, and for these definitions we can lay down the rule that

7 FREGE'S NOTE. I speak of parallelism here so as to express myself more conveniently and be more easily understood. The essentials of this discussion can easily be carried over to the case of identity of numbers.

8 FREGE'S NOTE. Non inelegans specimen demonstrandi in abstractis [A not inelegant example of proof in abstract matters]. Erdm. p. 94.

9 FREGE'S NOTE. In a hypothetical judgment, for example, an identity of directions could occur as the condition or as the consequence.

it must remain possible to replace the direction of a line by that of a line parallel to it.

66.1 But a third doubt arises about our attempted definition. In the proposition "the direction of a is identical with the direction of b", the direction of a appears as an object[10], and in our definition we have a means of recognizing this object again in case it should happen to appear in some other guise, say as the direction of b. But this means does not suffice for all cases. It does not allow us to decide, for example, whether England is the same as the direction of the earth's axis. Forgive this apparently nonsensical example! Of course no one will confuse England with the direction of the earth's axis; but that is no thanks to our definition. Our definition says nothing about whether the proposition "the direction of a is identical with q" is to be affirmed or denied unless q itself is given in the form "the direction of b". What we lack is the concept of direction. If we had it, we could stipulate: if q is not a direction, our proposition is to be denied; if q is a direction, the earlier definition decides. Now it is tempting to define: q is a direction if there is a line b whose direction is q. But then it is clear that we have gone round in a circle. To be able to apply this definition, we must already know in every case whether the proposition "q is identical with the direction of b" would have to be affirmed or denied.

67.1 Suppose someone wanted to say: q is a direction if it is introduced by means of the definition stated above. Then they would be treating the way in which the object q is introduced as a property of it, which it is not. The definition of an object, as such, really asserts nothing about it; it lays down the meaning of a sign. Once that has been done, it turns into a judgment that deals with the object. But then it no longer introduces the

10 FREGE'S NOTE. The definite article indicates this. For me a concept is a possible predicate of a singular judgeable content, and an object is a possible subject of one. If, in the proposition "the direction of the telescope's axis is identical with the direction of the earth's axis", we regard the direction of the telescope's axis as the subject, then the predicate is "identical with the direction of the earth's axis". This is a concept. But the direction of the earth's axis is only part of the predicate; it is an object, since it can also be made the subject.

object, and it stands on the same footing as other statements about it. If we chose this way out, we would be presupposing that an object can be given in only one single way. Otherwise it would not follow from the fact that q is not introduced by our definition that it could not be introduced in that way. All equations would then come down to this: that what is given to us in the same way is recognized as the same. But this is so self-evident and so unfruitful that it would not be worth stating. Indeed, one could draw no conclusion from it that differed from every one of the premises. Rather, the many-sided and significant applicability of equations rests on the fact that we can recognize something again even though it is given in different ways.

THE DEFINITION OF NUMBER

Frege defines the Number of a concept as the extension of the concept "equinumerous with" it.

68.1 Since in this way we cannot obtain a sharply delimited concept of direction, nor, for the same reasons, such a concept of Number, let us try another route. If line a is parallel to line b, then the extension of the concept "line parallel to line a" is identical with the extension of the concept "line parallel to line b". Conversely, if the extensions of these concepts are identical, then a is parallel to b. Let us try, then, to define: the direction of line a is the extension of the concept "parallel to line a"; the shape of triangle d is the extension of the concept "similar to triangle d".

68.2 If we want to apply this to our own case, we must put concepts in place of the lines or triangles. In place of parallelism or similarity we must put the possibility of correlating one-to-one the objects falling under the one concept with those falling under the other. For brevity I shall call the concept F *equinumerous* with the concept G if this possibility obtains. But I must ask that this word be regarded as an arbitrarily chosen notation,

whose meaning is to be gathered not from its linguistic composition but from this stipulation.[11]

68.3 Accordingly I define: the Number which belongs to the concept F is the extension[12] of the concept "equinumerous with the concept F".

69.1 That this definition is correct will perhaps not seem very evident at first. Does one not think of something else under the extension of a concept? What one thinks of under it becomes clear from the primary statements that can be made about extensions of concepts. They are the following: 1. identity; 2. that one is more comprehensive than the other.

69.2 Now the proposition "the extension of the concept 'equinumerous with the concept F' is identical with the extension of the concept 'equinumerous with the concept G'" is true always and only when the proposition "the same number belongs to the concept F as to the concept G" is also true. Here, then, there is complete agreement.

69.3 It is true that we do not say one number is more comprehensive than another in the sense in which the extension of one concept is more comprehensive than that of another. But the case where the extension of the concept "equinumerous with the concept F" is more comprehensive than the extension of the concept "equinumerous with the concept G" cannot arise at all. Rather, if all concepts equinumerous with G are also equinumerous with F, then conversely all concepts equinumerous with F are also equinumerous with G. This "more comprehensive" must of course not be confused with the "greater" that occurs with numbers.

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11 German "gleichzahlig", literally "equal-numbered"; hence Frege's warning not to read the meaning off the compound.

12 FREGE'S NOTE. I believe that for "extension of the concept" one could simply say "concept". But two objections would be raised. 1. This contradicts my earlier assertion that the individual number is an object. That is indicated by the definite article in expressions like "the two", by the impossibility of speaking of ones, twos, etc. in the plural, and by the fact that the number makes up only part of the predicate of the statement of number. 2. Concepts can have the same extension without coinciding. I do indeed hold that both objections can be met; but that would take us too far afield here. I assume that it is known what the extension of a concept is.

69.4 Admittedly, it is still conceivable that the extension of the concept "equinumerous with the concept F" should be more or less comprehensive than some other extension of a concept, which then, by our definition, could not be a Number. And it is not usual to call a Number more or less comprehensive than the extension of a concept. But nothing prevents us from adopting such a way of speaking, should this case ever arise.

TESTING THE DEFINITION

Equinumerosity is defined by one-to-one correlation, using only logical relations.

70.1 Definitions prove their worth by their fruitfulness. Those that could just as well be left out without opening a gap in the chain of proof are to be rejected as utterly worthless.

70.2 Let us see, then, whether familiar properties of numbers can be derived from our definition of the Number which belongs to the concept F. We shall content ourselves here with the simplest.

70.3 For this it is necessary to grasp equinumerosity somewhat more precisely. We defined it by means of one-to-one correlation. I must now set out how I want this expression to be understood, since one might easily suspect that it contains something intuitive.

70.4 Consider the following example. Suppose a waiter wants to be sure of laying exactly as many knives on the table as plates. He need count neither the one nor the other, provided he lays a knife immediately to the right of each plate, so that every knife on the table lies immediately to the right of a plate. The plates and knives are thus correlated one-to-one with each other, and indeed by the same spatial relationship. If we think of different objects being substituted again and again for α and A in the proposition "α lies immediately to the right of A", then the part of the content that remains unchanged throughout constitutes the essence of the relation. Let us generalize this.

70.5 If, from a judgeable content that deals with an object a and an object b, we separate off a and b, we are left with a relational concept, which accordingly needs completing in two ways. If, in the proposition "the earth has more mass than the moon", we separate off "the earth", we obtain the concept "having more mass than the moon". If, on the other hand, we separate off the object "the moon", we get the concept "having less mass than the earth". If we separate off both at once, what remains is a relational concept. By itself it has no more sense than a simple concept does: it always requires completion into a judgeable content. But this can happen in different ways: instead of earth and moon I can, for example, put sun and earth, and it is precisely by this that the separating off is effected.

70.6 The individual pairs of correlated objects stand to the relational concept—as its subjects, one might say—much as the individual object stands to the concept under which it falls. The subject here is a compound one. Sometimes, when the relation is a reversible one, this also finds linguistic expression, as in the sentence "Peleus and Thetis were the parents of Achilles"[13]. By contrast, it would hardly be possible, for example, to render the content of the sentence "the earth is larger than the moon" so that "the earth and the moon" appeared as a compound subject, because "and" always suggests a certain parity. But this is beside the point.

70.7 The relational concept thus belongs, like the simple one, to pure logic. What matters here is not the particular content of the relation but solely its logical form. And whatever can be asserted of this form is analytic in its truth and is known *a priori*. This holds of relational concepts as of the others.

70.8 Just as "a falls under the concept F" is the general form of a judgeable content that deals with an object a, so we can take "a stands in the relation ϕ to b" as the general form of a judgeable content that deals with the object a and the object b.

71.1 Now suppose every object that falls under the concept F stands in the relation ϕ to an object falling under the concept G,

and for every object that falls under G there is an object falling under F that stands in the relation ϕ to it. Then the objects falling under F and G are correlated with one another by the relation ϕ.

71.2 It may still be asked what the expression "every object that falls under F stands in the relation ϕ to an object falling under G" means if no object at all falls under F. I understand it thus: the two propositions "a falls under F" and "a stands in the relation ϕ to no object falling under G" cannot both hold, whatever a may denote, so that either the first or the second or both are false. From this it emerges that "every object that falls under F stands in the relation ϕ to an object falling under G" holds if there is no object falling under F, because then the first proposition, "a falls under F", is always to be denied, whatever a may be.

71.3 Similarly, "for every object that falls under G there is one falling under F that stands in the relation ϕ to it" means that the two propositions "a falls under G" and "no object falling under F stands in the relation ϕ to a" cannot both hold, whatever a may be.

72.1 We have now seen when the objects falling under the concepts F and G are correlated with one another by the relation ϕ. Here, however, this correlation is to be one-to-one. By this I understand that the following two propositions hold: 1. if d stands in the relation ϕ to a, and if d stands in the relation ϕ to e, then in general, whatever d, a and e may be, a is the same as e; 2. if d stands in the relation ϕ to a, and if b stands in the relation ϕ to a, then in general, whatever d, b and a may be, d is the same as b.

72.2 We have thereby reduced one-to-one correlation to purely logical relationships and can now define as follows:

72.3 let the expression "the concept F is equinumerous with the concept G" mean the same as the expression "there is a relation ϕ which correlates one-to-one the objects falling under the concept F with the objects falling under G". I repeat: the Number

13 FREGE'S NOTE. This is not to be confused with the case where "and" only appears to join the subjects but in truth joins two sentences.

which belongs to the concept F is the extension of the concept "equinumerous with the concept F", and I add:

72.4 let the expression "n is a Number" mean the same as the expression "there is a concept such that n is the Number which belongs to it".

72.5 Thus the concept of Number is defined—apparently, to be sure, in terms of itself, but nevertheless without error, since "the Number which belongs to the concept F" has already been defined.

73.1 We shall now first show that the Number which belongs to the concept F is identical with the Number which belongs to the concept G if the concept F is equinumerous with the concept G. This sounds, of course, like a tautology. But it is not one, since the meaning of the word "equinumerous" comes not from its composition but from the definition just given.

73.2 According to our definition, it must be shown that the extension of the concept "equinumerous with the concept F" is the same as the extension of the concept "equinumerous with the concept G" if the concept F is equinumerous with the concept G. In other words, it must be proved that, under this assumption, the following propositions hold generally: if the concept H is equinumerous with the concept F, then it is also equinumerous with the concept G; and if the concept H is equinumerous with the concept G, then it is also equinumerous with the concept F.

73.3 The first proposition comes down to this. There is a relation that correlates one-to-one the objects falling under the concept H with those falling under the concept G, if there is a relation ϕ that correlates one-to-one the objects falling under the concept F with those falling under the concept G, and if there is a relation ψ that correlates one-to-one the objects falling under the concept H with those falling under the concept F. The following arrangement of the letters will make this easier to survey: H ψ F ϕ G.

73.4 Such a relation can in fact be specified. It lies in the content "there is an object to which c stands in the relation ψ and

which stands in the relation ϕ to b", if we separate off c and b from it (regard them as the terms of the relation). It can be shown that this relation is one-to-one, and that it correlates the objects falling under the concept H with those falling under the concept G.

73.5 The other proposition can be proved in a similar way[14]. These hints will, I hope, show clearly enough that here we need take no ground of proof from intuition, and that something can be done with our definitions.

ZERO, ONE AND THE SUCCESSOR

Zero, one and the relation of following directly in the series of natural numbers are defined.

74.1 We can now pass on to the definitions of the individual numbers.

74.2 Since nothing falls under the concept "not identical with itself", I define: 0 is the Number which belongs to the concept "not identical with itself".

74.3 Some may take offense at my speaking of a concept here. It may be objected that it contains a contradiction, and our old acquaintances, wooden iron and the square circle, may be recalled. Well, I think they are not nearly as bad as they are made out to be. Useful they will hardly be; but neither can they do any harm, so long as one does not presuppose that something falls under them. And merely using the concepts does not yet do that. That a concept contains a contradiction is not always so obvious that no investigation is needed. For that, one must first have the concept and treat it logically just like any other.[15] All that can be demanded of a concept from the side of logic, and for the rigor of proof, is that it be sharply delimited: that it be determined for every object whether or not it falls under it. Now

14 FREGE'S NOTE. Likewise the converse: if the number which belongs to the concept F is the same as that which belongs to the concept G, then the concept F is equinumerous with the concept G.

15 "Wooden iron" (hölzernes Eisen) is the proverbial German example of a contradiction in terms.

concepts that contain a contradiction, such as "not identical with itself", fully satisfy this requirement; for we know of every object that it does not fall under such a concept[16].

74.4 I use the word "concept" in such a way that "a falls under the concept F" is the general form of a judgeable content that deals with an object a and remains judgeable whatever is put for a. And in this sense "a falls under the concept 'not identical with itself'" means the same as "a is not identical with itself" or "a is not identical with a".

74.5 I could have taken any other concept under which nothing falls for the definition of 0. But what mattered to me was to choose one of which this can be proved purely logically. For this, "not identical with itself" offers itself most conveniently, where for "identical" I adopt the definition of *Leibniz* cited earlier, which is purely logical.

75.1 It must now be possible to prove, by means of the earlier stipulations, that every concept under which nothing falls is equinumerous with every concept under which nothing falls, and only with such a concept. From this it follows that 0 is the Number which belongs to such a concept, and that no object falls under a concept if the number which belongs to it is 0.

75.2 Suppose that no object falls either under the concept F or under the concept G. Then, to prove equinumerosity, we need a relation ϕ of which the following propositions hold: every object that falls under F stands in the relation ϕ to an object that falls under G; for every object that falls under G there is one falling under F that stands in the relation ϕ to it.

16 FREGE'S NOTE. Quite different from this is the definition of an object in terms of a concept under which it falls. The expression "the greatest proper fraction", for example, has no content, because the definite article lays claim to point to a definite object. By contrast, the concept "fraction that is less than 1 and is such that no fraction less than 1 exceeds it in magnitude" is quite unobjectionable. Indeed, to be able to prove that there is no such fraction, one even needs this concept, although it contains a contradiction. But if one wanted to use this concept to determine an object falling under it, it would certainly be necessary first to show two things: 1. that an object falls under this concept; 2. that only a single object falls under it. Since the first of these propositions is already false, the expression "the greatest proper fraction" is senseless.

75.3 Given what was said earlier about the meaning of these expressions, under our assumptions every relation fulfills these conditions. So too, therefore, does identity, which is moreover one-to-one; for the two propositions required above for this hold of it.

75.4 If, on the other hand, an object falls under G, say a, while none falls under F, then the two propositions "a falls under G" and "no object falling under F stands in the relation ϕ to a" both hold, for every relation ϕ. The first is true by the first assumption and the second by the second: if there is no object falling under F, then there is also none that stands in any relation to a. There is therefore no relation which, according to our definition, correlates the objects falling under F with those falling under G, and accordingly the concepts F and G are not equinumerous.

76.1 I shall now define the relation in which any two adjacent members of the series of natural numbers stand to each other. Let the proposition "there is a concept F and an object x falling under it such that the Number which belongs to the concept F is n, and the Number which belongs to the concept 'falling under F but not identical with x' is m" mean the same as "n follows directly after m in the series of natural numbers".

76.2 I avoid the expression "n is *the* Number following next after m", because two propositions would first have to be proved to justify the definite article[17]. For the same reason I do not yet say here "n = m + 1"; for the identity sign, too, designates (m + 1) as an object.

77.1 Now, to arrive at the number 1, we must first show that there is something that follows directly after 0 in the series of natural numbers.

77.2 Consider the concept—or, if you prefer, the predicate —"identical with 0". 0 falls under it. Under the concept "identical with 0 but not identical with 0", on the other hand, no object falls, so that 0 is the Number which belongs to this concept. We therefore have a concept "identical with 0" and an object 0 fall-

17 FREGE'S NOTE. See the note on pp. 87 and 88.

ing under it, of which the following holds: the Number which belongs to the concept "identical with 0" is identical with the Number which belongs to the concept "identical with 0"; the Number which belongs to the concept "identical with 0 but not identical with 0" is 0.

77.3 Hence, according to our definition, the Number which belongs to the concept "identical with 0" follows directly after 0 in the series of natural numbers.

77.4 If we now define: 1 is the Number which belongs to the concept "identical with 0", then we can express the last proposition thus: 1 follows directly after 0 in the series of natural numbers.

77.5 It is perhaps not superfluous to remark that the definition of 1 does not presuppose any observed fact[18] for its objective legitimacy. This is easily confused with the fact that certain subjective conditions must be fulfilled to make the definition possible for us, and that sense perceptions prompt us to it[19]. That may well be so without the derived propositions ceasing to be a priori. Among such conditions is, for example, also that blood of sufficient quantity and the right quality should flow through the brain—at least so far as we know—but the truth of our last proposition is independent of this. It continues to hold even if this no longer happens; and even if all rational beings were at some point to fall into hibernation at the same time, its truth would not be suspended for that period but would remain quite undisturbed. The truth of a proposition is simply not its being thought.

78.1 Here I list some propositions that are to be proved by means of our definitions. The reader will easily see how this can be done. 1. If a follows directly after 0 in the series of natural numbers, then a = 1. 2. If 1 is the Number which belongs to a

18 FREGE'S NOTE. A proposition without generality.

19 FREGE'S NOTE. Cf. B. Erdmann, *Die Axiome der Geometrie* [The Axioms of Geometry], p. 164.

TRANSLATOR'S NOTE. Benno Erdmann (1851–1921), German philosopher; the book appeared in 1877.

concept, then there is an object that falls under the concept. 3. If 1 is the Number which belongs to a concept F; if the object x falls under the concept F, and if y falls under the concept F, then x = y; that is, x is the same as y. 4. If an object falls under a concept F, and if from the fact that x falls under the concept F and that y falls under the concept F it can in general be inferred that x = y, then 1 is the Number which belongs to the concept F. 5. The relation of m to n that is set up by the proposition "n follows directly after m in the series of natural numbers" is one-to-one. This does not yet say that for every Number there is another that follows directly after it, or after which it follows directly, in the series of numbers.

78.2 6. Every Number except 0 follows directly after a Number in the series of natural numbers.

79.1 Now, to be able to prove that after every Number (n) in the series of natural numbers some Number follows directly, one must exhibit a concept to which this latter Number belongs. We choose as such the concept "belonging to the series of natural numbers ending with n", which must first be defined.

FOLLOWING IN A SERIES

Frege defines following in a series and shows that the series of natural numbers has no end.

79.2 I shall first repeat, in somewhat different words, the definition of following in a series that I gave in my *Begriffsschrift*.[20]

79.3 Let the proposition "if every object to which x stands in the relation ϕ falls under the concept F, and if from the fact that d falls under the concept F it follows in general, whatever d may be, that every object to which d stands in the relation ϕ falls under the concept F, then y falls under the concept F, whatever concept F may be" mean the same as "y follows x in the ϕ-series" and as "x precedes y in the ϕ-series".

80.1 A few remarks on this will not be superfluous. Since the relation ϕ has been left undetermined, the series need not

necessarily be thought of in the form of a spatial or temporal arrangement, although these cases are not excluded.

80.2 One might perhaps think another definition more natural, for example: if, starting from x, we keep shifting our attention from one object to another to which it stands in the relation ϕ, and if in this way we can finally reach y, then we say that y follows x in the ϕ-series.

80.3 This is a way of investigating the matter, not a definition. Whether our wandering attention reaches y may depend on all sorts of subjective incidental circumstances, for example on the time at our disposal or on our knowledge of things. Whether y follows x in the ϕ-series has, in general, nothing whatever to do with our attention and the conditions of its movement. It is a matter of fact: just as a green leaf reflects certain light rays whether or not they fall into my eye and produce a sensation; just as a grain of salt is soluble in water whether or not I throw it into the water and observe the process; and just as it is still soluble even when I have no possibility at all of experimenting with it.

80.4 My definition raises the matter out of the realm of subjective possibilities into that of objective determinacy. Indeed, that one proposition follows from certain others is something objective, independent of the laws governing the movement of our attention, and it makes no difference to this whether we actually draw the inference or not. Here we have a characteristic mark that decides the question wherever it can be raised, even if in a particular case we may be prevented by external difficulties from judging whether it applies. That is irrelevant to the matter itself.

80.5 We do not always need to run through all the intermediate members from the initial member to an object in order to be certain that the object follows it. If, for example, we are given that in the ϕ-series b follows a and c follows b, then by our

20 Frege's *Begriffsschrift* ("concept-script", 1879) set out his logical notation; its third part defines following in a series.

definition we can infer that c follows a, without even knowing the intermediate members.

80.6 Only through this definition of following in a series does it become possible to reduce the mode of inference from n to (n + 1), which is seemingly peculiar to mathematics, to the general laws of logic.

81.1 Now if the relation φ we have is the one in which m is placed to n by the proposition "n follows directly after m in the series of natural numbers", then instead of "φ-series" we say "series of natural numbers".

81.2 I further define:

81.3 let the proposition "y follows x in the φ-series or y is the same as x" mean the same as "y belongs to the φ-series beginning with x" and as "x belongs to the φ-series ending with y".

81.4 Accordingly, a belongs to the series of natural numbers ending with n if n either follows a in the series of natural numbers or is identical with a[21].

82.1 It must now be shown that—under a condition still to be specified—the Number which belongs to the concept "belonging to the series of natural numbers ending with n" follows directly after n in the series of natural numbers. This then proves that there is a Number following directly after n in the series of natural numbers, and that there is no last member of this series. Obviously this proposition cannot be established empirically or by induction.

82.2 To give the proof itself here would take us too far. Only its course may be briefly indicated. It must be proved: 1. if a follows directly after d in the series of natural numbers, and if it holds of d that the Number which belongs to the concept "belonging to the series of natural numbers ending with d" follows directly after d in the series of natural numbers, then it also holds of a that the Number which belongs to the concept "belonging to the series of natural numbers ending with a" follows directly after a in the series of natural numbers.

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21 FREGE'S NOTE. If n is not a Number, then only n itself belongs to the series of natural numbers ending with n. Do not take offense at the expression!

82.3 Secondly, it must be proved that what is asserted of d and of a in the propositions just stated holds of 0; and it must then be inferred that it also holds of n, if n belongs to the series of natural numbers beginning with 0. This mode of inference is an application of the definition I have given of the expression "y follows x in the series of natural numbers", in which one has to take as the concept F that common statement about d and a, about 0 and n.

83.1 To prove proposition (1) of the preceding §, we must show that a is the Number which belongs to the concept "belonging to the series of natural numbers ending with a, but not identical with a". And for this it must in turn be proved that this concept has the same extension as the concept "belonging to the series of natural numbers ending with d". For this one needs the proposition that no object belonging to the series of natural numbers beginning with 0 can follow itself in the series of natural numbers. This too must be proved by means of our definition of following in a series, as indicated above[22].

83.2 We are thereby compelled to add a condition to the proposition that the Number which belongs to the concept "belonging to the series of natural numbers ending with n" follows directly after n in the series of natural numbers: namely, that n belong to the series of natural numbers beginning with 0. For this there is a shorter, customary way of speaking, which I now define: let the proposition "n belongs to the series of natural numbers beginning with 0" mean the same as "n is a finite Number".

83.3 Then we can express the last proposition thus: no finite Number follows itself in the series of natural numbers.

22 FREGE'S NOTE. E. Schröder, op. cit., p. 63, seems to regard this proposition as a consequence of a notation that could also be conceived otherwise. Here too we notice the drawback that impairs his whole presentation of this matter: one does not really know whether number is a sign, and in that case what its meaning is, or whether it is just this meaning. From the fact that one lays down different signs, so that the same one never recurs, it does not yet follow that these signs also mean different things.

INFINITE NUMBERS

Frege introduces an infinite Number and compares his account with Cantor's.

84.1 Opposed to the finite Numbers are the infinite ones. The Number which belongs to the concept "finite Number" is an infinite one. Let us denote it, say, by ∞_1! If it were finite, it could not follow itself in the series of natural numbers. But it can be shown that ∞_1 does so.

84.2 There is nothing in any way mysterious or wonderful about the infinite Number ∞_1 so defined. "The Number which belongs to the concept F is ∞_1" now means neither more nor less than this: there is a relation which correlates one-to-one the objects falling under the concept F with the finite Numbers. According to our definitions this has a perfectly clear and unambiguous sense; and that is enough to justify the use of the sign ∞_1 and to secure it a meaning. That we can form no idea of an infinite Number is quite irrelevant, and would apply equally to finite Numbers. In this way our Number ∞_1 is something just as determinate as any finite one: it can without doubt be recognized again as the same and distinguished from another.

85.1 Recently *G. Cantor* has introduced infinite Numbers in a remarkable work[23]. I wholly agree with his assessment of the view that would admit only the finite Numbers as actual at all. Neither these nor fractions, nor negative, irrational and complex numbers, are perceptible by the senses or spatial. If we call actual what acts on the senses, or at least has effects that can have sense perceptions as their nearer or more remote consequences, then of course none of these numbers is actual. But we have no need at all of such perceptions as grounds of proof for our theorems. A name or sign that has been introduced in a logically unobjectionable way we may use in our investigations

23 FREGE'S NOTE. *Grundlagen einer allgemeinen Mannichfaltigkeitslehre* [Foundations of a General Theory of Manifolds]. Leipzig, 1883.

without misgiving; and so our Number ∞_1 is as justified as two or three.

85.2 Although in this, I believe, I am in agreement with *Cantor*, I diverge from him somewhat in terminology. What I call Number he calls "power", while his concept[24] of Number refers to ordering. For finite Numbers, independence of order does indeed result after all, but not for infinitely large ones. Now the ordinary use of the word "Number" and of the question "how many?" contains no reference to any particular ordering. Cantor's Number rather answers the question: "at what position in the succession does the last member stand?" That is why my terminology seems to me to agree better with ordinary usage.[25] When one extends the meaning of a word, one must take care that as many general propositions as possible retain their validity, especially such fundamental ones as, for Number, its independence of order. We have needed no extension at all, because our concept of Number covers infinite numbers as well from the outset.

86.1 To obtain his infinite Numbers, *Cantor* introduces the relational concept of following in a succession, which differs from my "following in a series". On his account, a succession would arise, for example, if one arranged the finite positive integers so that the odd ones followed one another in their natural order among themselves, and likewise the even ones among themselves, and it were further stipulated that every even number should follow every odd one. In this succession, for example, 0 would follow 13. But no number would directly precede 0. Now this is a case that cannot occur in following in a series as I have defined it. It can be rigorously proved, without using any axiom of intuition, that if y follows x in the ϕ-series, there is an object that directly precedes y in this series. Now it seems to me that precise definitions of following in a succession and of Cantor's

24 FREGE'S NOTE. This expression may seem to contradict the objectivity of the concept emphasized earlier; but here only the naming is subjective.

25 Cantor's "Mächtigkeit" (power) is what is now called cardinality; his "Anzahl" is an ordinal notion.

Number are still lacking. So *Cantor* appeals to the somewhat mysterious "inner intuition" where a proof from definitions ought to be sought and would no doubt be possible. For I believe I can foresee how those concepts could be determined. In any case, I by no means wish these remarks to attack their legitimacy and fruitfulness. On the contrary, I welcome these investigations as an extension of science, especially because they open up a purely arithmetical route to higher infinitely large Numbers (powers).

V. Conclusion

Schluss

Arithmetic as developed logic; the fruitfulness of definitions; other numbers—negative, fractional, irrational and complex.

87.1 I hope I have made it probable in this work that the laws of arithmetic are analytic judgments and consequently a priori. Arithmetic would then be only a further developed logic, and every arithmetical proposition a law of logic, albeit a derived one. The applications of arithmetic to the explanation of nature would be logical treatments of observed facts[1]; calculating would be inferring. The laws of number will not, as *Baumann*[2] thinks, need practical confirmation in order to be applicable in the external world. For in the external world, the totality of what is spatial, there are no concepts, no properties of concepts, no numbers. So the laws of number are not really applicable to external things: they are not laws of nature. But they are applicable to judgments that hold of things in the external world: they are laws of the laws of nature. They assert a connection not between natural phenomena but between judgments; and among these are the laws of nature too.

1 FREGE'S NOTE. Observation itself already involves a logical activity.
2 FREGE'S NOTE. Op. cit., vol. II, p. 670.

KANT AND ANALYTIC JUDGMENTS

Fruitful definitions extend our knowledge though their consequences are analytic.

88.1 *Kant*[3] evidently underestimated the value of analytic judgments—probably as a result of defining the concept too narrowly—although the broader concept used here seems to have been in his mind[4]. If his definition is taken as the basis, the division into analytic and synthetic judgments is not exhaustive. He is thinking of the case of the universal affirmative judgment. There one can speak of a subject concept and ask whether the predicate concept is contained in it—by virtue of the definition. But what if the subject is an individual object? What if it is an existential judgment? Then there can be no talk at all of a subject concept in this sense. *Kant* seems to think of the concept as determined by coordinate characteristic marks; but that is one of the least fruitful ways of forming concepts. If one surveys the definitions given above, one will hardly find one of this kind. The same holds of the really fruitful definitions in mathematics, for example that of the continuity of a function. There we do not have a series of coordinate characteristic marks, but a more intimate, I would say more organic, connection of the determinations. The difference can be made intuitive by a geometrical picture. If the concepts (or their extensions) are represented by regions of a plane, then the concept defined by coordinate characteristic marks corresponds to the region common to all the regions of the marks; it is enclosed by parts of their boundaries. With such a definition, then, it is a matter—to speak figuratively—of using lines already given in a new way to mark off a region[5]. But nothing essentially new comes to light in this. The more fruitful definitions of concepts draw boundary lines that were

3 FREGE'S NOTE. Op. cit. III, pp. 39 ff.

4 FREGE'S NOTE. On p. 43 he says that a synthetic proposition can be grasped in accordance with the principle of contradiction only if another synthetic proposition is presupposed.

5 FREGE'S NOTE. Likewise when the characteristic marks are joined by "or".

not yet given at all. What can be inferred from them cannot be surveyed in advance; one is not simply taking out of the box again what one had put into it. These consequences extend our knowledge, and so, according to *Kant*, one ought to regard them as synthetic; yet they can be proved purely logically and are therefore analytic. They are in fact contained in the definitions, but like the plant in the seed, not like the beam in the house. Often several definitions are needed to prove a proposition, which is consequently contained in no single one of them and yet follows purely logically from all of them together.

89.1 I must also contradict the generality of *Kant's* assertion[6] that without sensibility no object would be given to us. Zero and one are objects that cannot be given to us through the senses. Even those who regard the smaller numbers as intuitable will have to concede that none of the numbers greater than 1000^(1000^1000) can be given to them in intuition, and that we nevertheless know a good deal about them. Perhaps *Kant* used the word "object" in a somewhat different sense. But then zero, one and our ∞_1 fall entirely outside his consideration, for they are not concepts either, and of concepts too *Kant*[1] demands that we attach to them their object in intuition.

89.2 I do not want to bring on myself the reproach of petty fault-finding toward a mind we can only look up to with grateful admiration. So I believe I must also stress our agreement, which far outweighs the disagreement. To touch only on what lies nearest here: I see a great merit of *Kant's* in his having drawn the distinction between synthetic and analytic judgments. In calling geometrical truths synthetic and a priori, he revealed their true nature. And this is still worth repeating today, because it is still often misunderstood. If *Kant* was mistaken about arithmetic, that does not, I think, substantially detract from his merit. What mattered to him was that there are synthetic judgments a priori. Whether they occur only in geometry or in arithmetic as well is of lesser importance.

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6 FREGE'S NOTE. Op. cit. III, p. 82.

THE NEED FOR A GAPLESS PROOF

Only a gapless chain of logical inferences could prove that arithmetic is analytic; hence the Begriffsschrift.

90.1 I do not claim to have made the analytic nature of arithmetical propositions more than probable. One can still doubt whether their proof can be carried out entirely from purely logical laws, and whether a ground of proof of some other kind might not slip in unnoticed somewhere. Nor is this misgiving fully dispelled by the indications I have given for the proof of some propositions. It can be removed only by a gapless chain of inferences, such that no step is taken that does not conform to one of a few modes of inference recognized as purely logical. Hardly a single proof has so far been carried out in this way. The mathematician is satisfied if every transition to a new judgment is evidently correct, without asking about the nature of this evidence, whether it is logical or intuitive. Such a step is often very complex and equivalent to several simple inferences, alongside which something from intuition may also flow in. One proceeds by leaps, and from this arises the apparently overabundant variety of modes of inference in mathematics. For the bigger the leaps, the more manifold the combinations of simple inferences and axioms of intuition they can stand for. Nevertheless such a transition is often immediately evident to us, without our becoming conscious of the intermediate steps. And since it does not present itself as one of the recognized logical modes of inference, we are at once ready to take this evidence for intuitive and the inferred truth for synthetic, even when its domain of validity obviously extends beyond what can be intuited.

90.2 In this way it is not possible to separate cleanly the synthetic, which rests on intuition, from the analytic. Nor does one succeed in this way in compiling, with certainty, a complete list of the axioms of intuition, such that every mathematical proof can be carried out from these axioms alone according to the logical laws.

91.1 The demand that all leaps in inference be avoided is therefore inescapable. That it is so hard to satisfy is due to the tediousness of proceeding step by step. Every proof that is even slightly involved threatens to become monstrously long. In addition, the excessive variety of logical forms that have taken shape in language makes it difficult to mark off a range of modes of inference that suffices for all cases and is easy to survey.

91.2 To lessen these drawbacks I devised my *Begriffsschrift*. It is meant to achieve greater brevity and perspicuity of expression and to move within a few fixed forms, in the manner of a calculation, so that no transition is permitted that does not conform to rules laid down once and for all[7]. No ground of proof can then creep in unnoticed. In this way, without borrowing a single axiom from intuition, I have proved a proposition[8] that one might at first glance take for a synthetic one. I will state it here as follows:

91.3 If the relation of every member of a series to the next one is single-valued, and if m and y follow x in this series, then y either precedes m in this series, or coincides with it, or follows m.

91.4 From this proof one can see that propositions which extend our knowledge can contain analytic judgments[9].

7 FREGE'S NOTE. It is meant, however, to be capable of expressing not only the logical form, as Boole's notation does, but also a content.

8 FREGE'S NOTE. *Begriffsschrift*, Halle a/S. 1879, p. 86, formula 133.

9 FREGE'S NOTE. This proof will still be found far too long-winded, a drawback that may perhaps seem to more than outweigh its almost absolute security against error or gaps. My aim at the time was to reduce everything to the smallest possible number of the simplest possible logical laws. As a result I used only a single mode of inference. But even then I pointed out in the preface, p. VII, that for further application it would be advisable to admit more modes of inference. This can be done without impairing the cogency of the chain of inferences, and in this way a considerable abbreviation can be achieved.

OTHER NUMBERS

Frege criticizes formal theories of negative, fractional and complex numbers.

92.1 So far we have confined our consideration to the Numbers. Let us now also cast a glance at the other kinds of number, and try to make what we have learned in the narrower field useful for this wider one!

92.2 To make clear the sense of the question about the possibility of a certain number, *Hankel*[10] says:

92.3 "Number today is no longer a thing, a substance existing self-subsistently outside the thinking subject and the objects that give rise to it, a self-subsistent principle, as it was, say, for the Pythagoreans. The question of existence can therefore only be referred to the thinking subject or to the objects thought, whose relations the numbers represent. Strictly speaking, the mathematician counts as impossible only what is logically impossible, i.e. what contradicts itself. That numbers impossible in this sense cannot be admitted requires no proof. But if the numbers in question are logically possible, and their concept is clearly and determinately defined and therefore free of contradiction, then that question can only come down to this: whether there is, in the domain of the real or of what is actual in intuition, of the actual, a substrate for them; whether there are objects in which the numbers, that is, the intellectual relations of the specific kind, make their appearance."

93.1 With the first sentence one may be in doubt whether, according to *Hankel*, the numbers exist in the thinking subject, or in the objects that give rise to them, or in both. In the spatial sense, at any rate, they are neither inside nor outside either the subject or any object. They are, however, outside the subject in the sense that they are not subjective. Each of us can feel only his own pain, his own pleasure, his own hunger, and can have only his own sensations of sound and color. Numbers, by con-

10 FREGE'S NOTE. Op. cit. pp. 6 and 7.

trast, can be common objects for many, and indeed they are exactly the same for all, not merely more or less similar inner states of different people. If *Hankel* wants to refer the question of existence to the thinking subject, he seems thereby to make it a psychological question, which it in no way is. Mathematics is not concerned with the nature of our mind, and how any psychological questions are answered must be a matter of complete indifference to it.

94.1 The claim that the mathematician counts as impossible only what contradicts itself must also be challenged. A concept is admissible even if its characteristic marks contain a contradiction; one must only not presuppose that something falls under it. But from the fact that a concept contains no contradiction it cannot yet be inferred that something falls under it. How, moreover, is one to prove that a concept contains no contradiction? This is by no means always obvious. From the fact that one sees no contradiction it does not follow that there is none, and the determinateness of the definition offers no guarantee of it. *Hankel* proves[11] that a finite complex number system of higher order than the ordinary one, subject to all the laws of addition and multiplication, contains a contradiction. But that has to be proved; one does not see it at once. Before this has been done, someone using such a number system might well arrive at wonderful results, whose justification would be no worse than the one *Hankel*[12] gives for the theorems on determinants by means of the alternating numbers. For who can guarantee that a hidden contradiction is not contained in their concept too? And even if one could rule out such a contradiction in general, for arbitrarily many alternating units, it still would not follow that there are such units. And it is precisely this that we need. Let us take as an example the 18th proposition of Book I of Euclid's *Elements*:

94.2 In every triangle the greater angle lies opposite the greater side.

11 FREGE'S NOTE. Op. cit. pp. 106 and 107.
12 FREGE'S NOTE. Op. cit. § 35.

94.3 To prove this, *Euclid* marks off on the greater side AC a segment AD equal to the smaller side AB, appealing in doing so to an earlier construction. The proof would collapse if there were no such point, and it is not enough that one discovers no contradiction in the concept "point on AC whose distance from A is equal to B". B is then joined to D. That there is such a straight line is likewise a proposition on which the proof relies.

95.1 Strictly, the consistency of a concept can presumably be established only by showing that something falls under it. The converse would be a mistake. *Hankel* falls into this mistake when he says, with reference to the equation $x + b = c$[13]:

95.2 "It is obvious that if $b > c$ there is no number x in the series 1, 2, 3, ... that solves the problem in question: subtraction is then *impossible*. Nothing prevents us, however, from *regarding* the difference $(c - b)$ in this case as a sign that solves the problem, and with which we are to operate exactly as if it were a numerical number from the series 1, 2, 3, ..."

95.3 Something does indeed prevent us from simply regarding $(2 - 3)$ as a sign that solves the problem: an empty sign does not solve the problem. Without a content it is only ink or printer's ink on paper; as such it has physical properties, but not the property of giving 2 when increased by 3. It would really not be a sign at all, and its use as one would be a logical mistake. Even in the case where $c > b$, it is not the sign ("$c - b$") that is the solution of the problem, but its content.

96.1 One could just as well say: among the numbers known so far there is none that satisfies both equations $x + 1 = 2$ and $x + 2 = 1$ at once; but nothing prevents us from introducing a sign that solves the problem. It will be objected: but the problem contains a contradiction. Certainly, if one demands a real or an ordinary complex number as the solution. But let us just extend our number system, let us create numbers that meet the requirements! Let us wait and see whether anyone can show us a contradiction! Who can know what is possible with these new numbers? We shall then, of course, not be able to maintain the uniqueness

13 FREGE'S NOTE. Op. cit. p. 5. Similarly E. Kossak, op. cit. p. 17, bottom.

of subtraction. But then we also have to give up the uniqueness of taking roots if we want to introduce the negative numbers, and the complex numbers make taking logarithms many-valued.

96.2 Let us also create numbers that allow divergent series to be summed! No! The mathematician cannot create things at will, any more than the geographer can; he too can only discover what is there, and name it.

96.3 This error afflicts the formal theory of fractions, of negative numbers and of complex numbers[14]. One demands that the familiar rules of calculation be preserved, as far as possible, for the numbers to be newly introduced, and derives general properties and relations from this. If one nowhere runs into a contradiction, one regards the introduction of the new numbers as justified, as if a contradiction could not still be hidden somewhere, and as if freedom from contradiction were already existence.

97.1 That this mistake is so easily made is probably due to an inadequate distinction between concepts and objects. Nothing prevents us from using the concept "square root of -1". But we are not straightaway entitled to put the definite article in front of it and to regard the expression "the square root of -1" as meaningful. On the assumption that $i^2 = -1$, we can prove the formula that expresses the sine of a multiple of the angle α in terms of the sine and cosine of α itself. But we must not forget that the proposition then carries with it the condition $i^2 = -1$, which we may not simply drop. If there were nothing at all whose square was -1, the equation would not need to be correct by virtue of our proof[15], because the condition $i^2 = -1$, on which its validity appears to depend, would never be fulfilled. It would be as if we had used, in a geometrical proof, an auxiliary line that cannot be drawn at all.

98.1 *Hankel*[16] introduces two kinds of operation, which he calls lytic and thetic, and which he determines by certain properties

14 FREGE'S NOTE. Things are similar with *Cantor's* infinite Numbers.

15 FREGE'S NOTE. It might nonetheless be possible to prove it rigorously by some other route.

these operations are supposed to have. There is nothing to be said against this, as long as one does not presuppose that there are such operations, and objects that can be their results[17]. Later[18] he denotes a thetic, completely unique, associative operation by (a + b), and the corresponding, likewise completely unique, lytic operation by (a − b). Such an operation? Which one? Any one at all? Then this is no definition of (a + b). And what if there is none?[19] If the word "addition" had no meaning yet, it would be logically admissible to say: we will call such an operation an addition. But one may not say: such an operation is to be called the addition and denoted by (a + b), before it is established that there is one and only one. One may not use the indefinite article on one side of a defining equation and the definite article on the other. Then *Hankel* says, without further ado, "the modulus of the operation", without having proved that there is one and only one.

99.1 In short, this purely formal theory is inadequate. What is valuable in it is only this. One proves that if operations have certain properties, such as associativity and commutativity, certain propositions hold of them. One then shows that addition and multiplication, which are already known, have these properties, and can at once state those propositions of them without laboriously repeating the proof in each individual case. Only through this application to operations given in some other way does one arrive at the familiar propositions of arithmetic. But one must by no means believe that addition and multiplication can be introduced in this way. One gives only a guide for the definitions, not the definitions themselves. It is said: the name "addition" is to be given only to a thetic, completely unique, associative operation. But this does not yet specify at all which operation is now to bear this name. On this account nothing would

16 FREGE'S NOTE. Op. cit. p. 18.

17 FREGE'S NOTE. Hankel really does this already by using the equation Θ (c, b) = a.

18 FREGE'S NOTE. Op. cit. p. 29.

19 Hankel's terms, from Greek: "thetic" (positing) for direct operations such as addition, "lytic" (dissolving) for their inverses such as subtraction.

stand in the way of calling multiplication addition and denoting it by (a + b), and no one could say with certainty whether 2 + 3 was 5 or 6.

100.1 If we give up this purely formal approach, a way may seem to present itself in the fact that the meaning of the words "sum" and "product" is extended at the same time as new numbers are introduced. One takes an object, say the moon, and declares: the moon multiplied by itself is to be -1. Then in the moon we have a square root of -1. This definition seems permissible, because the sense of such a product does not yet follow at all from the previous meaning of multiplication, and can therefore be stipulated at will when this meaning is extended. But we also need the products of a real number with the square root of -1. So let us rather choose the time interval of one second as a square root of -1, and denote it by i! Then by 3 i we shall understand the time interval of 3 seconds, and so on.[20] What object, then, shall we denote by, say, 2 + 3i? What meaning would have to be given to the plus sign in this case? Well, that must be stipulated in general, which will admittedly not be easy. But let us suppose that we had secured a sense for all signs of the form a + bi, and indeed one such that the familiar laws of addition hold! Then we should further have to stipulate that in general (a + bi) (c + di) = ac − bd + i (ad + bc), thereby determining multiplication further.

101.1 Now we could prove the formula for cos (nα) if we knew that the identity of the real parts follows from the identity of complex numbers. That would have to emerge from the sense of a + bi, which we have here assumed to exist. The proof would hold only for the sense of complex numbers, of their sums and products, that we have stipulated. Now, since for integral real n and real α, i no longer occurs in the equation at all, one is temp-

20 FREGE'S NOTE. With the same right we could also choose a certain quantity of electricity, a certain area, etc., as square roots of -1, but would then of course have to denote these different roots differently. That one can apparently create as many square roots of -1 as one likes in this way becomes less surprising when one considers that the meaning of the square root was not already unalterably fixed before these stipulations, but is only partly determined by them.

ted to conclude: so it is quite immaterial whether i means a second, a millimeter or anything else, as long as our laws of addition and multiplication hold; they alone matter; we need not worry about the rest. Perhaps the meaning of a + bi, of sum and of product can be stipulated in various ways such that those laws remain in force. But it is not immaterial whether one can find any such sense for these expressions at all.

102.1 People often act as if the mere demand were already its fulfillment. One demands that subtraction[21], division and the extraction of roots always be performable, and believes that this is enough. Why does one not also demand that a straight line be drawn through any three points? Why does one not demand that all the laws of addition and multiplication hold for a three-dimensional complex number system as they do for the real one? Because this demand contains a contradiction. Well then, let one first prove that those other demands contain no contradiction! Until this has been done, all the much-coveted rigor is nothing but empty show and vapor.

102.2 The auxiliary line that may have been drawn for a proof does not occur in the geometrical theorem itself. Perhaps several such lines are possible, e.g. if a point can be chosen arbitrarily. But however dispensable each individual line may be, the force of the proof still depends on its being possible to draw a line of the required kind. The mere demand is not enough. So in our case too it is not immaterial to the force of the proof whether "a + bi" has a sense or is mere printer's ink. For this it is not sufficient to require that it should have a sense, or to say that its sense is the sum of a and bi, if one has not first explained what "sum" means in this case and has not justified the use of the definite article.

103.1 Admittedly, a good deal can be objected to our attempted stipulation of the sense of "i". We thereby bring something quite alien, namely time, into arithmetic. The second stands in no inner relation whatsoever to the real numbers. The propositions proved by means of complex numbers would be judgments a

21 FREGE'S NOTE. Cf. Kossak op. cit. p. 17.

posteriori, or at any rate synthetic, if there were no other kind of proof, or if no other sense could be found for i. At all events, the attempt must first be made to show that all propositions of arithmetic are analytic.

103.2 When *Kossak*[22] says with reference to the complex number:

103.3 "It is the composite idea of heterogeneous groups of mutually identical elements[23]", he seems thereby to have avoided bringing in anything alien. But he only seems to, and only because the expression is vague. We get no answer at all as to what 1 + i really means: the idea of an apple and a pear, or that of toothache and gout? It cannot mean both at once, since then 1 + i would not always be identical with 1 + i. It will be said: that depends on the particular stipulation. Well then, in *Kossak's* sentence too we still have no definition of the complex number at all, but only a general guide to one. But we need more: we must know definitely what "i" means. And if, following that guide, we were now to say: the idea of a pear, we would again be introducing something alien into arithmetic.

103.4 What is usually called the geometrical representation of complex numbers at least has this advantage over the attempts considered so far: in it 1 and i do not appear wholly unconnected and heterogeneous. Rather, the segment regarded as the representation of i stands in a law-governed relation to the segment by which 1 is represented. Strictly speaking, incidentally, it is not correct that here 1 means a certain segment and i a segment of equal length perpendicular to it; rather, "1" means the same thing everywhere. A complex number here indicates how the segment that counts as its representation arises from a given segment (the unit segment) by taking multiples, dividing and rotating[24]. But even on this account, every theorem whose proof

22 FREGE'S NOTE. Op. cit. p. 17.
23 FREGE'S NOTE. On the expression "idea" compare § 27; on "group", what was said about "aggregate" in § 23 and § 25; on the identity of the elements, §§ 34–39.
24 FREGE'S NOTE. For simplicity's sake I disregard incommensurables here.

must rely on the existence of a complex number appears dependent on geometrical intuition, and hence synthetic.

104.1 How, then, are fractions, irrational numbers and complex numbers to be given to us? If we enlist the help of intuition, we bring something alien into arithmetic. But if we merely determine the concept of such a number by characteristic marks, if we merely require that the number have certain properties, then nothing guarantees that anything actually falls under the concept and meets our requirements; and yet it is precisely on this that proofs must rely.

104.2 Well, how do things stand with Number? Are we really not allowed to speak of 1000^(1000^1000) until that many objects have been given to us in intuition? Is it an empty sign until then? No! It has a perfectly determinate sense, even though it is psychologically impossible, if only because of the shortness of our lives, to bring that many objects before our consciousness[25]. All the same, 1000^(1000^1000) is an object whose properties we can come to know, although it is not intuitable. One convinces oneself of this by showing, when introducing the sign a^n for the power, that it always expresses one and only one positive whole number if a and n are positive whole numbers. To set out in detail how this can be done would take us too far afield here. The way in which we defined zero in § 74, one in § 77, and the infinite Number ∞_1 in § 84, together with the sketch of the proof that in the series of natural numbers a Number follows directly after every finite Number (§§ 82 and 83), will show the general path.

104.3 In the end, with the definition of fractions, complex numbers and so on too, everything will come down to seeking out a judgeable content that can be transformed into an equation whose sides are then precisely the new numbers. In other words, we must fix the sense of a recognition judgment for such numbers. In doing so we must heed the misgivings about such a transformation that we discussed in §§ 63–68. If we proceed as

25 FREGE'S NOTE. A rough calculation shows that millions of years would be nowhere near enough for this.

we did there, the new numbers will be given to us as extensions of concepts.

105.1 This conception of numbers[26] easily explains, it seems to me, the fascination that working on arithmetic and analysis exerts. Adapting a well-known saying, one might well say: the proper object of reason is reason. In arithmetic we are concerned with objects that become known to us not as something alien, from outside, through the medium of the senses, but which are given directly to reason, which can see through them completely as what is most its own[27].[28]

105.2 And yet, or rather precisely for this reason, these objects are not subjective figments of the brain. There is nothing more objective than the laws of arithmetic.

SUMMARY OF THE ARGUMENT

Frege retraces the course of the whole inquiry and its results.

106.1 Let us take a brief look back over the course of our inquiry! We established that number is neither a heap of things nor a property of such a heap, but that it is also not a subjective product of mental processes; rather, a statement of number asserts something objective about a concept. Having established this, we first tried to define the individual numbers 0, 1 and so on, and the advance in the series of numbers. The first attempt failed, because we had defined only that assertion about concepts, and not 0 and 1 separately, which are only parts of it. The consequence was that we could not prove the identity of numbers. It turned out that the number with which arithmetic is concerned must be taken not as an attribute lacking self-subsist-

26 FREGE'S NOTE. One might also call it formal. But it is quite different from the theory assessed above under that name.

27 FREGE'S NOTE. I do not at all wish to deny by this that without sense impressions we would be as thick as a plank and would know nothing of numbers or of anything else. But this psychological proposition does not concern us here at all. Because of the constant danger of confusing two fundamentally different questions, I stress this once more.

28 Adapts Alexander Pope's line "The proper study of mankind is man" (*An Essay on Man*, 1734).

ence, but substantivally[29]. Number thus appeared as a recognizable object, though not as a physical or even merely spatial one, nor as one of which we can form a picture through the imagination. We then laid down the principle that the meaning of a word is to be explained not in isolation but in the context of a sentence. Only by adhering to it, I believe, can the physical conception of number be avoided without falling into the psychological one. Now there is a kind of proposition that must have a sense for every object: the recognition propositions, which in the case of numbers are called equations. The statement of number too, we saw, is to be construed as an equation. So the task was to fix the sense of a numerical identity, and to express it without making use of the number words or of the word "number". We recognized the possibility of correlating one-to-one the objects falling under a concept F with those falling under a concept G as the content of a recognition judgment for numbers. Our definition therefore had to present that possibility as equivalent in meaning to a numerical identity. We recalled similar cases: the definition of direction from parallelism, of shape from similarity, and so on.

107.1 The question then arose: when is one entitled to construe a content as that of a recognition judgment? For this the condition must be fulfilled that in every judgment the left-hand side of the tentatively assumed equation can be replaced by the right-hand side without affecting its truth. Now, to begin with, without the addition of further definitions, we know no statement about the left or right side of such an equation other than precisely that of identity. So substitutability needed to be demonstrated only in an equation.

107.2 But one misgiving still remained. A recognition proposition must always have a sense. Suppose we construe the possibility of correlating one-to-one the objects falling under the concept F with those falling under the concept G as an equation, by saying instead: "the Number which belongs to the concept F

29 FREGE'S NOTE. The difference corresponds to that between "blue" and "the color of the sky".

is identical with the Number which belongs to the concept G", and thereby introduce the expression "the Number which belongs to the concept F". Then we have a sense for the equation only if both sides have the form just mentioned. With such a definition we could not judge whether an equation is true or false if only one of its sides has this form. This led us to the definition:

107.3 The Number which belongs to the concept F is the extension of the concept "concept equinumerous with the concept F", where we called a concept F equinumerous with a concept G if that possibility of one-to-one correlation obtains.

107.4 Here we presupposed that the sense of the expression "extension of the concept" is known. This way of overcoming the difficulty will probably not meet with approval everywhere, and some will prefer to remove that misgiving in another way. I too attach no decisive importance to bringing in the extension of a concept.

108.1 It then remained to define one-to-one correlation; we reduced it to purely logical relations. We indicated the proof of the proposition: the number which belongs to the concept F is identical with the number which belongs to the concept G if the concept F is equinumerous with the concept G. We then defined 0, the expression "n follows directly after m in the series of natural numbers", and the number 1, and showed that 1 follows directly after 0 in the series of natural numbers. We cited some propositions that can easily be proved at this point, and then went into somewhat more detail on the following one, which reveals the infinity of the series of numbers:

108.2 In the series of natural numbers, every number is followed by a number.

108.3 This led us to the concept "belonging to the series of natural numbers ending with n", of which we wanted to show that the Number belonging to it follows directly after n in the series of natural numbers. We first defined it by means of the following of an object y after an object x in a general ϕ-series. The sense of this expression too was reduced to purely logical relations. In

this way we succeeded in showing that the mode of inference from n to (n + 1), which is usually taken to be peculiarly mathematical, rests on the general logical modes of inference.

108.4 To prove the infinity of the series of numbers, we then needed the proposition that no finite number follows itself in the series of natural numbers. We thus arrived at the concepts of finite and infinite number. We showed that the latter is fundamentally no less justified logically than the former. For comparison we brought in *Cantor's* infinite Numbers and his "following in the succession", pointing out the difference in terminology.

109.1 From all that preceded, it then emerged with great probability that arithmetical truths are analytic and a priori in nature; and we arrived at an improvement on Kant's view. We saw, further, what is still lacking to raise that probability to certainty, and indicated the path that must lead there.

109.2 Finally we used our results to criticize a formal theory of negative, fractional, irrational and complex numbers, and this criticism made its inadequacy evident. We saw its error in two things: it took the consistency of a concept to be proved if no contradiction had shown itself, and it treated the consistency of a concept as a sufficient guarantee that the concept is satisfied. This theory imagines that it need only make demands, and that their fulfillment then goes without saying. It behaves like a god who can create by his mere word whatever he needs. We also had to censure the passing off of an instruction for a definition as the definition itself: an instruction which, if followed, would bring something alien into arithmetic, although in its wording it can keep itself free of this, but only because it remains a mere instruction.

109.3 Thus that formal theory is in danger of falling back into the a posteriori, or at least the synthetic, however much it gives itself the air of floating on the heights of abstraction.

109.4 Our earlier treatment of the positive whole numbers has now shown us that it is possible to avoid the intrusion of external things and geometrical intuitions without falling into the error of that formal theory. As there, the point is to fix the

content of a recognition judgment. If we imagine this done everywhere, then the negative, fractional, irrational and complex numbers appear no more mysterious than the positive whole numbers, and these no more real, actual or tangible than those.