On Sense and Reference

GOTTLOB FREGE

1848–1925

On Sense and Reference

Über Sinn und Bedeutung

Translated from the German with AI (Claude) and edited



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On Sense and Reference · Gottlob Frege (1848–1925)

Translated from the German with AI (Claude) and edited. Reading edition, Forage, October 2026. forage.com/frege/sense-and-reference

German text: the first printing, *Zeitschrift für Philosophie und philosophische Kritik*, N. F., vol. 100 (Leipzig: C. E. M. Pfeffer, 1892), pp. 25–50, from the transcription of the Deutsches Textarchiv (www.deutschestextarchiv.de/frege_sinn_1892) (text public domain; transcription CC BY-SA 4.0), with the long s modernized. Frege's letter-spaced emphasis is shown in italics; his footnotes are kept, numbered through the essay. English: translated from the German with AI (Claude), then edited for accuracy and consistency. Not reviewed by a human specialist. Free to reuse under CC BY-SA 4.0. The small numbers set in the text are the page numbers of the 1892 journal and the paragraph beginning on that page (27.2 = p. 27, second paragraph); page breaks within a paragraph are marked in the text.

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CONTENTS

Introduction vii

On Sense and Reference 1

Sense and idea 5

The sense and reference of a sentence 8

The True and the False 10

Subordinate clauses 12

Clauses that name an object 16

Subsidiary thoughts 23

Back to identity 26

INTRODUCTION

Gottlob Frege (1848–1925) taught mathematics at Jena and spent his life trying to show that arithmetic is a branch of logic. Along the way he had to ask what words and sentences actually do, and in 1892 he published his answer in a philosophy journal: a twenty-six-page essay that became the starting point of modern philosophy of language.

The essay opens with a puzzle about identity. "The morning star is the morning star" is trivial, but "the morning star is the evening star" records an astronomical discovery—yet both names stand for the same planet, Venus. Frege's solution is to distinguish two things a sign has: its *reference*, the object it stands for, and its *sense*, the way that object is presented. Two names can share a reference and differ in sense; a name such as "Odysseus" can have a sense and perhaps no reference at all. Both are kept apart from the private ideas and images a word calls up in each of us.

Frege then carries the distinction from names to whole sentences. The sense of a sentence is the thought it expresses; its reference, he argues, is its truth value—the True or the False. Most of the essay tests this bold claim against subordinate clauses, reported speech and belief, conditionals and clauses of reason, before returning to the identity puzzle with which it began.

On Sense and Reference

Über Sinn und Bedeutung
Identity and the morning star; sense, reference and idea; the thought as the sense of a sentence and the truth value as its reference; subordinate clauses and indirect speech.

25.1 Identity[1] challenges reflection through questions that are bound up with it and are not altogether easy to answer. Is it a relation? A relation between objects? Or between names or signs for objects? The last is what I assumed in my *Begriffsschrift*. The reasons that seem to speak in its favor are the following. a=a and a=b are evidently sentences of different cognitive value. a=a holds a priori and, following Kant, is to be called analytic, whereas sentences of the form a=b often contain very valuable extensions of our knowledge and cannot always be established a priori. The discovery that it is not a new sun that rises every morning, but always the same one, was surely one of the most consequential in astronomy.[2] Even now, recognizing a small planet or a comet as one already observed is not always something 26 self-evident. Now if we wanted to see in identity a relation between what the names "a" and "b" refer to, it would seem that a=b could not differ from a=a—provided, that is, that a=b is true. A relation of a thing to itself would be expressed by it, and indeed a relation in which each thing stands to itself but no thing to any other. What one wants to say with a=b seems to be

1 FREGE'S NOTE. I use this word in the sense of identity, and understand "a=b" in the sense of "a is the same as b" or "a and b coincide".
2 Frege's *Begriffsschrift* ("Concept-Script", 1879) introduced his formal logical notation; there he treated identity as a relation between names.

that the signs or names “a” and “b” refer to the same thing. In that case it would be precisely those signs that were being talked about; a relation between them would be asserted. But this relation would hold between the names or signs only insofar as they name or designate something. It would be a mediated relation, arising through the connection of each of the two signs with the same thing designated. But this connection is arbitrary. Nobody can be forbidden to adopt any arbitrarily producible event or object as a sign for anything whatever. A sentence a=b would then no longer concern the matter itself, but only our manner of designation; we would not be expressing any genuine knowledge in it. But in many cases that is just what we want to do. Suppose the sign “a” differed from the sign “b” only as an object (here, by its shape), not as a sign—that is to say, not in the way it designates something. Then the cognitive value of a=a would be essentially equal to that of a=b, provided a=b is true. A difference can arise only if the difference between the signs corresponds to a difference in the mode of presentation of what is designated. Let a, b, c be the straight lines connecting the vertices of a triangle with the midpoints of the opposite sides. The point of intersection of a and b is then the same as the point of intersection of b and c. So we have different designations for the same point, and these names (“point of intersection of a and b”, “point of intersection of b and c”) at the same time indicate the mode of presentation; and that is why the sentence contains real knowledge.

26.1 It is natural, then, to think that connected with a sign (name, combination of words, written character) there is, besides what is designated—which may be called the reference of the sign—also what I should like to call the sense of the sign, in which the mode of presentation is contained. Accordingly, in our example the 27 reference of the expressions “the point of intersection of a and b” and “the point of intersection of b and c” would be the same, but not their sense. The reference of “even-

ing star" and "morning star" would be the same, but not the sense.[3]

27.1 It is clear from the context that by "sign" and "name" I have here understood any designation that stands in for a proper name, and whose reference is thus a definite object (this word taken in its widest extent)—but not a concept or a relation, which will be discussed more closely in another essay. The designation of a single object can also consist of several words or other signs. For brevity, every such designation may be called a proper name.[4]

27.2 The sense of a proper name is grasped by everyone who is sufficiently familiar with the language, or the totality of designations, to which it belongs[5]; but this only ever illuminates the reference, if there is one, from a single side. A comprehensive knowledge of the reference would require that we be able to say at once of any given sense whether it belongs to that reference. We never get that far.

27.3 The regular connection between a sign, its sense and its reference is such that to the sign there corresponds a definite sense, and to that sense in turn a definite reference, while to a reference (an object) there belongs more than one sign. The same sense has different expressions in different languages, and indeed even in the same language. To be sure, there are exceptions to this regular behavior. Certainly in a perfect totality of signs a definite sense ought to correspond to every expression; but ordinary languages [28] often fail to meet this requirement,

3 Frege's word *Bedeutung* is ordinary German for "meaning"; he makes it a technical term for what a sign stands for, rendered here as "reference". The evening star and the morning star are both the planet Venus.

4 The essay is Frege's "On Concept and Object" ("Über Begriff und Gegenstand"), also published in 1892.

5 FREGE'S NOTE. In the case of a genuine proper name such as "Aristotle", opinions about the sense may of course differ. One might, for instance, take it to be: the pupil of Plato and teacher of Alexander the Great. Anyone who does so will attach a different sense to the sentence "Aristotle was born in Stagira" from someone who took the sense of this name to be: the teacher of Alexander the Great who was born in Stagira. So long as the reference remains the same, these fluctuations of sense can be tolerated, although they too are to be avoided in the theoretical structure of a demonstrative science and ought not to occur in a perfect language.

and one must be content if the same word always has the same sense at least within the same context. Perhaps it may be granted that a grammatically well-formed expression that stands for a proper name always has a sense. But this does not settle whether a reference also corresponds to the sense. The words "the celestial body most distant from the earth" have a sense; but whether they also have a reference is very doubtful. The expression "the least convergent series" has a sense; but it can be proved to have no reference, since for every convergent series one can find a less convergent, but still convergent, one. So by grasping a sense one does not yet have a reference with certainty.

28.1 When words are used in the ordinary way, what one wants to talk about is their reference. But it can also happen that one wants to talk about the words themselves or about their sense. The former happens, for example, when one quotes another person's words in direct speech. One's own words then refer in the first place to the other person's words, and only these have the customary reference. We then have signs of signs. In writing, the word-images are in this case enclosed in quotation marks. So a word-image standing in quotation marks must not be taken in its customary reference.

28.2 If one wants to talk about the sense of an expression 'A', one can do so simply by means of the phrase "the sense of the expression 'A'". In indirect speech one talks about the sense of, for example, another person's remarks. It is clear from this that in this mode of speech, too, the words do not have their customary reference but refer to what is customarily their sense. To have a short expression, we will say: in indirect speech words are used *indirectly*, or have their *indirect* reference. Accordingly, we distinguish the *customary* reference of a word from its *indirect* reference, and its *customary* sense from its *indirect* sense. The indirect reference of a word is, then, its customary sense. Such exceptions must always be kept in view if one wants to grasp correctly how sign, sense and reference are connected in a particular case.

SENSE AND IDEA

The private idea a sign calls up is distinguished from its shared sense, illustrated by a telescope.

29.1 The reference and the sense of a sign are to be distinguished from the idea associated with it. If the reference of a sign is an object perceptible by the senses, my idea of it is an inner image[6] arising from memories of sense impressions I have had and of activities, inner as well as outer, that I have performed. This image is often steeped in feelings; the clarity of its separate parts varies and fluctuates. The same idea is not always connected with the same sense, even in the same person. The idea is subjective: one person's idea is not another's. This of itself gives rise to manifold differences among the ideas associated with the same sense. A painter, a rider and a zoologist will probably connect very different ideas with the name "Bucephalus".[7] This is what essentially distinguishes the idea from the sense of a sign, which can be the common property of many and is therefore not a part or a mode of the individual mind; for it can hardly be denied that humankind has a common store of thoughts, which it passes on from one generation to another[8].

29.2 So while there is no objection to speaking of the sense without qualification, in the case of the idea one must, strictly speaking, add to whom it belongs and at what time. One might perhaps say: just as one person connects this idea with the same word and another person that one, so too one person may associate this sense with it and another person that one. But then the difference lies only in the manner of this association. That does

6 FREGE'S NOTE. Along with ideas we can at once include intuitions, in which the sense impressions and activities themselves take the place of the traces they have left in the mind. The difference is unimportant for our purpose, especially since, alongside the sensations and activities, memories of such sensations and activities probably always help to complete the intuitive image. But one can also understand by intuition an object, insofar as it is perceptible by the senses or spatial.

7 Bucephalus was the horse of Alexander the Great.

8 FREGE'S NOTE. That is why it is inexpedient to designate such fundamentally different things by the word "idea" [Vorstellung].

not prevent both from grasping the same sense; [30] but they cannot have the same idea. *Si duo idem faciunt, non est idem* [If two people do the same thing, it is not the same thing]. When two people form an idea of the same thing, each still has their own idea. It is indeed sometimes possible to establish differences between the ideas, and even the sensations, of different people; but an exact comparison is not possible, because we cannot have these ideas together in the same consciousness.

[30.1] The reference of a proper name is the object itself that we designate by it; the idea we have in doing so is wholly subjective; in between lies the sense, which is indeed no longer subjective like the idea, but is not the object itself either. The following analogy may perhaps serve to make these relationships clear. Someone observes the moon through a telescope. I compare the moon itself with the reference; it is the object of the observation, which is mediated by the real image projected by the objective lens inside the telescope, and by the retinal image of the observer. The former I compare with the sense, the latter with the idea or intuition. The image in the telescope is indeed only one-sided; it depends on the observer's position; but it is nonetheless objective, insofar as it can serve several observers. It could at any rate be arranged for several people to use it at the same time. But each of them would still have a retinal image of their own. Even geometrical congruence would hardly be attainable, owing to the different structure of people's eyes, and actual coincidence would be out of the question. This analogy might perhaps be developed further by supposing that A's retinal image could be made visible to B; or that A himself could see his own retinal image in a mirror. This would perhaps show how an idea can indeed itself be taken as an object, but is nevertheless, as such, not for the observer what it is immediately for the person who has the idea. But pursuing this would probably lead us too far afield.

[30.2] We can now recognize three levels of difference between words, expressions and whole sentences. Either the difference concerns at most the ideas; or the sense but not the reference;

or, finally, the reference as well. With regard to [31] the first level, it should be noted that, owing to the uncertain connection of ideas with words, a difference may exist for one person that another does not find. The difference between a translation and the original text should properly not go beyond the first level. Among the differences still possible here are the colorings and illuminations that poetry and eloquence seek to give to the sense. These colorings and illuminations are not objective; every listener and reader must create them for themselves, following the hints of the poet or speaker. Without some affinity in the way human beings form ideas, art would of course not be possible; but how far the poet's intentions are fulfilled can never be exactly determined.

[31.1] In what follows there will be no further talk of ideas and intuitions; they have been mentioned here only so that the idea a word arouses in a hearer is not confused with its sense or its reference.

[31.2] To make brief and exact expression possible, let the following phrasings be laid down:

[31.3] A proper name (word, sign, combination of signs, expression) expresses its sense, refers to or designates its reference. By means of a sign we express its sense and designate its reference.

[31.4] Perhaps it has long since been objected, from the idealist and skeptical side: "You speak here without further ado of the moon as an object; but how do you know that the name 'the moon' has any reference at all? How do you know that anything at all has a reference?" I reply that it is not our intention to speak of our idea of the moon, and that we are not content with the sense either when we say 'the moon'; rather, we presuppose a reference. It would be to miss the sense altogether to suppose that in the sentence "the moon is smaller than the earth" an idea of the moon is being talked about. If the speaker intended that, he would use the phrase "my idea of the moon". Now we can of course be mistaken in this presupposition, and such mistakes have indeed occurred. But the question whether we are perhaps always mistaken in it can [32] be left unanswered here. To justify

speaking of the reference of a sign, it is enough for the moment to point to our intention in speaking or thinking—albeit with the proviso: if there is such a reference.

THE SENSE AND REFERENCE OF A SENTENCE

The thought a sentence expresses is its sense; the search for its reference begins with Odysseus.

32.1 So far we have considered sense and reference only for those expressions, words and signs that we have called proper names. We now ask about the sense and reference of a whole assertoric sentence. Such a sentence contains a thought[9]. Is this thought to be regarded as its sense or as its reference? Let us suppose for once that the sentence has a reference! If we now replace one word in it by another with the same reference but a different sense, this can have no effect on the reference of the sentence. But we see that in such a case the thought changes; for, to take an example,[10] the thought of the sentence "the morning star is a body illuminated by the sun" is different from that of the sentence "the evening star is a body illuminated by the sun". Someone who did not know that the evening star is the morning star might take the one thought to be true and the other to be false. So the thought cannot be the reference of the sentence; rather, we shall have to regard it as the sense. But what, then, about the reference? May we ask about it at all? Does a sentence as a whole perhaps have only a sense but no reference? One can in any case expect such sentences to occur, just as there are parts of sentences that have a sense but no reference. And sentences that contain proper names without reference will be of this kind. The sentence "Odysseus was set ashore at Ithaca while sound asleep" obviously has a sense. But since it is doubtful whether the name "Odysseus" occurring in it has a reference, it

9 FREGE'S NOTE. By a thought I understand not the subjective activity of thinking but its objective content, which is capable of being the common property of many.
10 By "thought" (*Gedanke*) Frege means not the act of thinking but its objective content, as his footnote [5] explains.

is thereby also doubtful whether the whole sentence has one. Yet it is certain that anyone who seriously takes the sentence to be true or false also ascribes to the name "Odysseus" a reference, not just a sense; for it is of the reference of this [33] name that the predicate is affirmed or denied. Whoever does not acknowledge a reference can neither affirm nor deny a predicate of it.[11] But advancing as far as the reference of the name would then be superfluous; one could be content with the sense if one wanted to stop at the thought. If all that mattered were the sense of the sentence, the thought, it would be unnecessary to concern oneself with the reference of a part of the sentence; for only the sense of that part, not its reference, can be relevant to the sense of the sentence. The thought remains the same whether the name "Odysseus" has a reference or not. That we take the trouble at all over the reference of a part of a sentence is a sign that we also acknowledge and demand, in general, a reference for the sentence itself. The thought loses value for us as soon as we recognize that one of its parts lacks reference. We are therefore surely entitled not to be content with the sense of a sentence, but to ask about its reference as well. But why, then, do we want every proper name to have not only a sense but also a reference? Why is the thought not enough for us? Because, and to the extent that, we are concerned with its truth value. This is not always the case. In listening to an epic poem, for instance, what captivates us, apart from the euphony of the language, is solely the sense of the sentences and the ideas and feelings they arouse. With the question of truth we would leave aesthetic enjoyment behind and turn to scientific investigation. That is also why it is a matter of indifference to us whether the name "Odysseus", for instance, has a reference, so long as we receive the poem as a work of art[12]. It is the striving for truth, then, that drives us always to advance from the sense to the reference.

11 Odysseus is the hero of Homer's *Odyssey*; in Book 13 the Phaeacians set him ashore on Ithaca while he sleeps.

12 FREGE'S NOTE. It would be desirable to have a special term for signs that are meant to have only a sense. If we call them, say, images, then the words of an actor on the stage would be images; indeed, the actor himself would be an image.

33.1 We have seen that a reference is to be sought for a sentence whenever the reference of its components matters; and that is the case whenever, and only when, we ask about the truth value.

THE TRUE AND THE FALSE

Frege proposes that the reference of a sentence is its truth value, either the True or the False.

34.1 We are thus driven to accept the *truth value* of a sentence as its reference. By the truth value of a sentence I understand the circumstance that it is true or that it is false. There are no further truth values. For brevity I call the one the True, the other the False. Every assertoric sentence in which the reference of the words matters is therefore to be regarded as a proper name, and its reference, if it has one, is either the True or the False. These two objects are acknowledged, if only tacitly, by everyone who judges at all, who takes anything to be true, and so even by the skeptic. Calling truth values objects may here still look like an arbitrary whim and perhaps a mere play on words, from which no far-reaching conclusions ought to be drawn. What I call an object can be discussed more exactly only in connection with concept and relation. I shall reserve that for another essay. But this much should already be clear here: in every judgment[13] —however obvious it may be—the step from the level of thoughts to the level of references (the objective) has already been taken.

34.2 One might be tempted to regard the relation of the thought to the True not as that of sense to reference but as that of subject to predicate. After all, one can say outright: "the thought that 5 is a prime number is true". But if one looks more closely, one notices that this really says nothing more than the simple sentence "5 is a prime number". In both cases the assertion of truth lies in the form of the assertoric sentence; and where this form does not have its usual force, for instance in the mouth of an

13 FREGE'S NOTE. For me a judgment is not the mere grasping of a thought, but the acknowledgment of its truth.

actor on the stage, the sentence "the thought that 5 is a prime number is true" likewise contains only a thought, and indeed the same thought as the simple "5 is a prime number". From this it may be gathered that the relation of the thought to the True must not, after all, be compared with that of subject to predicate. [35] Subject and predicate (understood in the logical sense) are, after all, parts of a thought; they stand at the same level for cognition. By putting subject and predicate together one only ever arrives at a thought; one never gets from a sense to its reference, never from a thought to its truth value. One moves at the same level, but one does not advance from one level to the next. A truth value cannot be part of a thought, any more than, say, the sun can, because it is not a sense but an object.

[35.1] If our supposition that the reference of a sentence is its truth value is correct, then the truth value must remain unchanged when a part of the sentence is replaced by an expression with the same reference but a different sense. And that is indeed the case. Leibniz declares outright: "Eadem sunt, quae sibi mutuo substitui possunt, salva veritate" [Those things are the same which can be substituted for one another without loss of truth]. And what else but the truth value could be found that belongs quite generally to every sentence in which the reference of the components matters at all, and that would remain unchanged under a replacement of the kind indicated?[14]

[35.2] Now if the truth value of a sentence is its reference, then on the one hand all true sentences have the same reference, and on the other hand so do all false ones. We see from this that in the reference of a sentence everything specific is obliterated. So we can never be concerned with the reference of a sentence alone. But neither does the mere thought yield knowledge; only the thought together with its reference, that is, its truth value, does. Judging can be regarded as advancing from a thought to its truth value. Of course this is not meant as a definition. Judging is something quite peculiar and incomparable. One might also say

14 Gottfried Wilhelm Leibniz (1646–1716), German philosopher and mathematician; the principle comes from his logical writings.

that judging is distinguishing parts within the truth value. This distinguishing takes place by going back to the thought. Every sense belonging to a truth value would correspond to its own way of analyzing it. I have, admittedly, used the word "part" here in a special way. That is, I have transferred the relation of whole and part from the sentence to its reference, by calling the reference of a word part of the reference of the sentence when the word itself is [36] part of that sentence. This way of speaking is of course open to objection, because in the case of reference the whole and one part do not determine the other part, and because the word "part" is already used of bodies in another sense. A special expression would have to be created for this.

36.1 The supposition that the truth value of a sentence is its reference shall now be tested further. We have found that the truth value of a sentence remains untouched when we replace an expression in it by another with the same reference; but we have not yet considered the case where the expression to be replaced is itself a sentence. Now if our view is correct, the truth value of a sentence containing another sentence as a part must remain unchanged when we substitute for the component sentence another whose truth value is the same. Exceptions are to be expected when the whole or the component sentence is direct or indirect speech; for, as we have seen, the reference of the words is then not the customary one. In direct speech a sentence refers to a sentence in turn, and in indirect speech to a thought.

SUBORDINATE CLAUSES

Frege tests his view on subordinate clauses, starting with indirect speech and reported belief.

36.2 We are thus led to consider subordinate clauses. These occur as parts of a complex sentence, which from the logical point of view likewise appears as a sentence, namely as a main clause. But here we are confronted with the question whether it also holds of subordinate clauses that their reference is a truth value.

Of indirect speech we already know the opposite. Grammarians regard subordinate clauses as representatives of parts of sentences, and divide them accordingly into noun clauses, relative (adjectival) clauses and adverbial clauses. From this one might derive the supposition that the reference of a subordinate clause is not a truth value but is of the same kind as that of a noun or an adjective or an adverb—in short, of a part of a sentence whose sense is not a thought but only part of one. Only a more thorough investigation can bring clarity here. In it we shall not keep strictly to the grammatical guideline, but shall group together what is logically of the same kind. Let us first look for cases in which the sense of the subordinate clause, as we have just supposed, is not an independent thought.

37.1 Indirect speech, too, belongs among the abstract noun clauses introduced by "that" [daß]. In it, as we have seen, the words have their indirect reference, which coincides with what is customarily their sense. In this case, then, the subordinate clause has as its reference a thought, not a truth value; and as its sense not a thought, but the sense of the words "the thought that …", which is only part of the thought of the whole complex sentence. This occurs after "say", "hear", "think", "be convinced", "conclude" and similar words.[15] The situation is different, and indeed rather complicated, after words like "recognize", "know", "imagine", which will be considered later.

37.2 That in our cases the reference of the subordinate clause is indeed the thought can also be seen from the fact that it is a matter of indifference to the truth of the whole whether that thought is true or false. Compare, for example, the two sentences "Copernicus believed that the orbits of the planets were circles" and "Copernicus believed that the apparent motion of the sun was produced by the real motion of the earth". Here one subordinate clause can be substituted for the other without harm to the truth. The main clause together with the subordin-

15 FREGE'S NOTE. In "A lied that he had seen B" the subordinate clause refers to a thought of which it is said, first, that A asserted it as true and, secondly, that A was convinced of its falsity.

ate clause has as its sense only a single thought, and the truth of the whole implies neither the truth nor the untruth of the subordinate clause. In these cases it is not permissible to replace an expression in the subordinate clause by another that has the same customary reference, but only by one that has the same indirect reference, that is, the same customary sense. Suppose someone wanted to conclude: the reference of a sentence is not its truth value, "for then it could be replaced everywhere by another with the same truth value". He would be proving too much; one could just as well maintain that the reference of the word "morning star" is not Venus, since one may not say "Venus" in place of "morning star" everywhere. The only legitimate conclusion is that the reference of a sentence is not always its truth value, and that "morning star" does not [38] always refer to the planet Venus—namely, not when this word has its indirect reference. Such an exceptional case occurs in the subordinate clauses just considered, whose reference is a thought.

[38.1] If one says "it seems that …", one means "it seems to me that …" or "I think that …". So we have the same case again. Matters are similar with expressions such as "be glad", "regret", "approve", "blame", "hope", "fear". If Wellington, toward the end of the battle of Belle-Alliance, was glad that the Prussians were coming, the ground of his joy was a conviction. Had he been deceived, he would have been no less glad for as long as his illusion lasted; and before he became convinced that the Prussians were coming, he could not be glad about it, even though they were in fact already approaching.[16]

[38.2] Just as a conviction or a belief can be the ground of a feeling, it can also be the ground of a conviction, as in drawing conclusions. In the sentence "Columbus concluded from the roundness of the earth that he could reach India by traveling west", we have, as the references of parts, two thoughts: that the earth is round, and that Columbus could reach India by traveling west. Again, all that matters here is that Columbus was convinced of

16 Belle-Alliance is the Prussian name for the battle of Waterloo (18 June 1815), where Blücher's Prussians arrived late in the day to support Wellington.

the one and of the other, and that the one conviction was the ground of the other. Whether the earth is really round, and whether Columbus could really reach India by traveling west, as he thought, is a matter of indifference to the truth of our sentence. But it is not a matter of indifference whether we put, in place of "the earth", "the planet accompanied by a moon whose diameter is greater than a quarter of its own". Here too we have the indirect reference of the words.

38.3 Adverbial clauses of purpose with "so that" [damit] also belong here; for evidently the purpose is a thought. Hence: indirect reference of the words, subjunctive [Conjunctiv].

38.4 The subordinate clause with "that" [daß] after "command", "request", "forbid" would appear in direct speech as an imperative. Such an imperative has no reference, but only a sense. A command or a request is not, indeed, a thought, but it stands at the same level as thoughts. Hence in subordinate clauses depending on "command", [39] "request", etc., the words have their indirect reference. The reference of such a clause is therefore not a truth value but a command, a request and the like.

39.1 It is similar with the dependent question in phrases such as "doubt whether", "not know what". That here too the words are to be taken in their indirect reference is easy to see. Dependent interrogative clauses with "who" [wer], "what" [was], "where" [wo], "when" [wann], "how" [wie], "by what means" [wodurch], etc., sometimes appear to come very close to adverbial clauses, in which the words have their customary reference. Linguistically, these cases are distinguished by the mood of the verb. With the subjunctive [Conjunctiv] we have a dependent question and indirect reference of the words, so that a proper name cannot in general be replaced by another name of the same object.

39.2 In the cases considered so far, the words in the subordinate clause had their indirect reference, and this made it understandable that the reference of the subordinate clause itself was also indirect—that is, not a truth value but a thought, a command, a request, a question. The subordinate clause could be

regarded as a noun; indeed, one might say, as a proper name of that thought, that command, etc., in which role it entered into the context of the complex sentence.

CLAUSES THAT NAME AN OBJECT

Clauses such as "he who discovered the elliptic form of the planetary orbits" refer to an object, here Kepler.

39.3 We now come to other subordinate clauses, in which the words do have their customary reference, yet without a thought appearing as the sense and a truth value as the reference. How this is possible is best made clear by examples.

39.4 "He who discovered the elliptic form of the planetary orbits died in misery."

39.5 If the subordinate clause here had a thought as its sense, it would have to be possible to express this thought in a main clause too. But this cannot be done, because the grammatical subject "he who" [der] has no independent sense, but mediates the relations to the consequent (main) clause "died in misery". Hence the sense of the subordinate clause is not a complete thought either, and its reference is not a truth value but Kepler. It might be objected that the sense of the whole does nevertheless contain a thought as a part, namely that there was someone who first recognized the elliptic form of the planetary orbits; for whoever takes the whole to be true 40 cannot deny this part. The latter is beyond doubt; but only because otherwise the subordinate clause "he who discovered the elliptic form of the planetary orbits" would have no reference.[17] When one asserts something, there is always the obvious presupposition that the simple or compound proper names used have a reference. So if one asserts "Kepler died in misery", it is presupposed that the name "Kepler" designates something; but this does not mean that the thought that the name "Kepler" designates something is contained in the sense of the sentence "Kepler died in misery".

17 Johannes Kepler (1571–1630), German astronomer, discovered that the planets move in elliptical orbits; he died in poverty.

If that were the case, the negation would have to run not "Kepler did not die in misery" but "Kepler did not die in misery, or the name 'Kepler' is without reference". Rather, that the name "Kepler" designates something is a presupposition just as much for the assertion "Kepler died in misery" as for its opposite. Now languages have the defect that expressions are possible in them which, by their grammatical form, seem destined to designate an object, but in particular cases fail to fulfill this destiny, because that depends on the truth of a sentence. Thus whether the subordinate clause "he who discovered the elliptic form of the planetary orbits" really designates an object, or only gives the appearance of doing so while in fact being without reference, depends on the truth of the sentence "there was someone who discovered the elliptic form of the planetary orbits". And so it may seem as if our subordinate clause contained, as part of its sense, the thought that there was someone who discovered the elliptic form of the planetary orbits. If that were right, the negation would have to run: "He who first recognized the elliptic form of the planetary orbits did not die in misery, or there was no one who discovered the elliptic form of the planetary orbits." [41] So this is due to an imperfection of language, from which, incidentally, even the symbolic language of analysis is not entirely free; there too combinations of signs can occur that give the appearance of referring to something but that, at least so far, are still without reference—for example, divergent infinite series. This can be avoided, for example, by the special stipulation that divergent infinite series are to refer to the number 0. Of a logically perfect language (*Begriffsschrift*) it must be demanded that every expression formed as a proper name, in a grammatically correct way, from signs already introduced should also in fact designate an object, and that no sign should be newly introduced as a proper name without a reference being secured for it. Logic textbooks warn against the ambiguity of expressions as a source of logical errors. I consider it at least as appropriate to warn against apparent proper names that have no reference. The history of mathematics has tales to tell of errors that have

arisen from this. Demagogic abuse lies just as close at hand here as with ambiguous words—perhaps closer. "The will of the people" can serve as an example; for it will be easy to establish that there is at any rate no generally accepted reference for this expression. It is therefore by no means unimportant to stop up the source of these errors once and for all, at least for science. Then objections such as the one just discussed will become impossible, because it can then never depend on the truth of a thought whether a proper name has a reference.

41.1 In our discussion we can take up, after these noun clauses, a kind of relative clause and adverbial clause that is logically closely related to them.

41.2 Relative clauses, too, serve to form compound proper names, even if, unlike noun clauses, they are not sufficient for this on their own. These relative clauses are to be regarded as equivalent to adjectives. Instead of "the square root of 4 which is less than 0" one can also say "the negative square root of 4". Here we have the case in which a compound proper name is formed from a concept-expression with the help of the definite article in the singular. This is permissible at any rate when one object, 42 and only one, falls under the concept[18]. Now concept-expressions can be formed in such a way that characteristic marks are specified by relative clauses, as in our example by the clause "which is less than 0". It is evident that such a relative clause, no more than the noun clause before it, can have a thought as its sense or a truth value as its reference. Its sense is only part of a thought, which in some cases can also be expressed by a single adjective. Here too, as with those noun clauses, there is no independent subject, and hence no possibility of rendering the sense of the subordinate clause in an independent main clause.

42.1 Places, instants and spans of time are, logically considered, objects; hence the linguistic designation of a particular

place, a particular instant or a particular span of time is to be regarded as a proper name. Now adverbial clauses of place and time can be used to form such a proper name in a way similar to what we have just seen with noun clauses and relative clauses. In the same way, expressions can be formed for concepts under which places, etc., fall. Here too it should be noted that the sense of these subordinate clauses cannot be rendered in a main clause. An essential component is missing, namely the specification of place or time, which is merely indicated by a relative pronoun or a conjunction[19].

42.2 In conditional clauses, too, as we have just 43 seen with noun clauses, relative clauses and adverbial clauses, one must usually recognize an indefinitely indicating component, to which a similar component corresponds in the consequent clause. Because each points to the other, they combine the two clauses into a whole that, as a rule, expresses only one thought. In the sentence "If a number is less than 1 and greater than 0, its

18 FREGE'S NOTE. In accordance with what was remarked above, such an expression ought really always to be assured of a reference by a special stipulation—for example, by laying down that its reference is to count as the number 0 when no object, or more than one, falls under the concept.

19 FREGE'S NOTE. Different interpretations of these sentences are, incidentally, easily possible. The sense of the sentence "After Schleswig-Holstein was separated from Denmark, Prussia and Austria quarreled" can also be rendered in the form "After the separation of Schleswig-Holstein from Denmark, Prussia and Austria quarreled". In this version it is surely clear enough that what is to be taken as part of this sense is not the thought that Schleswig-Holstein was at some time separated from Denmark. Rather, this is the necessary presupposition for the expression "after the separation of Schleswig-Holstein from Denmark" to have any reference at all. Our sentence can, of course, also be interpreted as saying that Schleswig-Holstein was at some time separated from Denmark. Then we have a case that will be considered later. To see the difference more clearly, let us put ourselves in the mind of a Chinese person who, knowing little of European history, believes it false that Schleswig-Holstein was ever separated from Denmark. Taking our sentence in the first way, he will regard it as neither true nor false, but will deny it any reference, because its subordinate clause would lack one. That clause would only seem to give a specification of time. If, on the other hand, he takes our sentence in the second way, he will find expressed in it a thought that he would hold to be false, alongside a part that would, for him, be without reference.

TRANSLATOR'S NOTE. Denmark lost the duchies of Schleswig and Holstein to Prussia and Austria in the war of 1864. Their quarrel over the territory led to the Austro-Prussian War of 1866.

square is also less than 1 and greater than 0", this component is "a number" in the conditional clause and "its" in the consequent clause. It is precisely through this indefiniteness that the sense acquires the generality one expects of a law. But precisely this also has the effect that the conditional clause on its own has no complete thought as its sense. Together with the consequent clause it expresses one thought, and indeed only a single one, whose parts are no longer thoughts. It is in general incorrect to say that in a hypothetical judgment two judgments are put into reciprocal relation. If one says this, or something like it, one is using the word "judgment" in the same sense that I have attached to the word "thought", so that I would say instead: "in a hypothetical thought two thoughts are put into reciprocal relation." This could be true only if an indefinitely indicating component were absent[20]; but then there would be no generality either.

43.1 When a point in time is to be indicated indefinitely in the conditional and consequent clauses, this is often done merely by the present tense of the verb, which in this case does not also designate the present. This grammatical form is then the indefinitely indicating component in the main clause and the subordinate clause. "When 44 the sun is in the Tropic of Cancer, we have the longest day in the northern hemisphere" is an example. Here too it is impossible to express the sense of the subordinate clause in a main clause, because this sense is not a complete thought. For if we said "the sun is in the Tropic of Cancer", we would be relating this to our present, and thereby changing the sense. Nor is the sense of the main clause a thought; only the whole consisting of main clause and subordinate clause contains one. Several common components in the conditional and consequent clauses may, incidentally, also be indicated indefinitely.

44.1 It is evident that noun clauses with "who" [wer] or "what" [was], and adverbial clauses with "where" [wo], "when" [wann],

20 FREGE'S NOTE. Sometimes there is no explicit linguistic indication, and it has to be gathered from the whole context.

"wherever" [wo immer], "whenever" [wann immer], are often to be understood, as far as their sense goes, as conditional clauses; for example, "Whoever touches pitch defiles himself."[21]

44.2 Relative clauses, too, can stand in for conditional clauses. Thus we can also express the sense of the sentence cited earlier in the form "the square of a number that is less than 1 and greater than 0 is less than 1 and greater than 0".

44.3 The matter is quite different when the common component of main clause and subordinate clause is designated by a proper name. In the sentence "Napoleon, who recognized the danger to his right flank, himself led his guards against the enemy position" two thoughts are expressed: 1. Napoleon recognized the danger to his right flank; 2. Napoleon himself led his guards against the enemy position. When and where this happened can, it is true, be known only from the context, but it is to be regarded as determined by it. If we utter our whole sentence as an assertion, we thereby assert both component sentences at once. If one of these component sentences is false, the whole is thereby false. Here we have the case in which the subordinate clause on its own has a complete thought as its sense (if we supplement it with a specification of time and place). Accordingly, the reference of the subordinate clause is a truth value. We may therefore expect that it can be replaced, without detriment to the truth of the whole, by a sentence with 45 the same truth value. This is indeed the case. Only, it must be noted that its subject has to be "Napoleon", for a purely grammatical reason: only then can it be put in the form of a relative clause attached to "Napoleon". But if one waives the demand that it appear in this form, and also allows it to be joined on with "and", this restriction falls away.

45.1 Complete thoughts are also expressed in subordinate clauses with "although" [obgleich]. This conjunction really has no sense, nor does it change the sense of the sentence; it only illuminates it in a peculiar way[22]. We could, it is true, replace the

21 A proverb from the Book of Sirach (Ecclesiasticus) 13:1.

22 FREGE'S NOTE. We have something similar with "but" [aber] and "yet" [doch].

concessive clause by another of the same truth value without detriment to the truth of the whole; but the illumination would then easily seem inappropriate, as if one were to sing a song with sad content to a cheerful tune.

45.2 In the last cases the truth of the whole included the truth of the component sentences. It is otherwise when a conditional clause expresses a complete thought by containing, instead of the merely indicating component, a proper name or something to be regarded as equivalent to one. In the sentence "If the sun has already risen now, the sky is heavily overcast", the time is the present, and thus determinate. The place, too, is to be thought of as determinate. Here one can say that a relation has been laid down between the truth values of the conditional clause and the consequent clause, namely that the case does not occur in which the conditional clause refers to the True and the consequent clause to the False. Accordingly, our sentence is true both if the sun has not yet risen now, whether the sky is heavily overcast or not, and if the sun has already risen and the sky is heavily overcast. Since only the truth values matter here, each of the component sentences can be replaced by another of the same truth value without changing the truth value of the whole. Here too, of course, the illumination would mostly become inappropriate; the thought might easily seem 46 absurd; but that has nothing to do with its truth value. One must always bear in mind here that subsidiary thoughts resonate along with it. These are not, however, actually expressed, and so must not be counted as part of the sense of the sentence; their truth value, therefore, cannot matter[23].

46.1 With this, the simple cases may be considered dealt with. Let us look back here at what we have found.

46.2 The subordinate clause usually has as its sense not a thought but only part of one, and consequently has as its reference no truth value. The reason for this is either that in the sub-

23 FREGE'S NOTE. The thought of our sentence could also be expressed thus: "Either the sun has not yet risen now, or the sky is heavily overcast", from which it can be seen how this kind of sentence connection is to be understood.

ordinate clause the words have their indirect reference, so that the reference, not the sense, of the subordinate clause is a thought; or that the subordinate clause is incomplete because it contains a merely indefinitely indicating component, so that it expresses a thought only together with the main clause. But there are also cases in which the sense of the subordinate clause is a complete thought; and then it can be replaced, without detriment to the truth of the whole, by another of the same truth value, insofar as there are no grammatical obstacles.

SUBSIDIARY THOUGHTS

Some clauses express more than one thought, so they cannot be swapped for others with the same truth value.

46.3 If one examines in this light all the subordinate clauses one comes across, one will soon meet some that will not fit properly into these pigeonholes. The reason, as far as I can see, will be that these subordinate clauses do not have so simple a sense. Almost always, it seems, we connect with a main thought that we utter subsidiary thoughts which the hearer, too, links with our words in accordance with psychological laws, although they are not expressed. And because they seem to be connected with our words of their own accord, almost like the main thought itself, we may then also want to express such a subsidiary thought along with it. In this way the sense of the sentence becomes richer, and it may well happen that we have more simple thoughts than sentences. In some cases the sentence must be understood in this way; in others it may be doubtful whether the subsidiary thought belongs to the sense of the sentence or 47 merely accompanies it[24]. Thus one might perhaps find that the sentence "Napoleon, who recognized the danger to his right flank, himself led his guards against the enemy position" expresses not only the two thoughts given above, but also the thought that his recognition of the danger was the reason why

24 FREGE'S NOTE. This can become important for the question whether an assertion is a lie, or an oath perjury.

he led the guards against the enemy position. One may in fact be in doubt whether this thought is merely lightly suggested or really expressed. Let us put to ourselves the question whether our sentence would be false if Napoleon had already made his decision before perceiving the danger. If our sentence could be true in spite of this, our subsidiary thought should not be regarded as part of the sense of our sentence. One will probably decide in favor of this. Otherwise the situation would become quite complicated: we would then have more simple thoughts than sentences. If we now also replaced the sentence "Napoleon recognized the danger to his right flank" by another of the same truth value, for example by "Napoleon was already over 45 years old", this would change not only our first thought but also our third, and so the truth value of the latter could also become different—namely, if his age was not the reason for his decision to lead the guards against the enemy. From this it can be seen why, in such cases, sentences with the same truth value cannot always take each other's place. The sentence then expresses, precisely by virtue of its connection with another, more than it does on its own.

47.1 Let us now consider cases where this happens regularly. In the sentence "Bebel fancies [wähnt] that France's desire for revenge can be appeased by the return of Alsace-Lorraine" two thoughts are expressed, but it is not the case that one belongs to the main clause and the other to the subordinate clause. They are: 1) Bebel believes that France's desire for revenge can be appeased by the return of Alsace-Lorraine; [48] 2) France's desire for revenge cannot be appeased by the return of Alsace-Lorraine.[25]

48.1 In the expression of the first thought the words of the subordinate clause have their indirect reference, while the same words in the expression of the second thought have their customary reference. We see from this that the subordinate clause in our original complex sentence is really to be taken twice,

25 August Bebel (1840–1913), leader of the German Social Democrats. Alsace-Lorraine was annexed by Germany from France in 1871.

with different references, one of which is a thought, the other a truth value. Now since the truth value is not the whole reference of the subordinate clause, we cannot simply replace it by another of the same truth value. We have something similar with expressions like "know", "recognize", "it is known".

48.2 With a subordinate clause of reason and the associated main clause we express several thoughts, which do not, however, correspond to the clauses one by one. In the sentence "Because ice is less dense than water, it floats on water" we have: 1) ice is less dense than water; 2) if something is less dense than water, it floats on water; 3) ice floats on water.

48.3 The third thought need not, perhaps, be stated explicitly, since it is contained in the first two. On the other hand, neither the first and third together, nor the second and third together, would make up the sense of our sentence. One now sees that our subordinate clause "because ice is less dense than water" expresses both our first thought and part of our second. That is why we cannot simply replace our subordinate clause by another of the same truth value: this would also change our second thought, and that could easily affect its truth value too.

48.4 The situation is similar in the sentence "If iron were [wäre] less dense than water, it would float on water." [49] Here we have the two thoughts that iron is not less dense than water, and that something floats on water if it is less dense than water. The subordinate clause again expresses one of these thoughts and part of the other.

49.1 Suppose we take the sentence considered earlier, "After Schleswig-Holstein was separated from Denmark, Prussia and Austria quarreled", in such a way that it expresses the thought that Schleswig-Holstein was at some time separated from Denmark. Then we have, first, this thought, and second, the thought that Prussia and Austria quarreled at a time more precisely determined by the subordinate clause. Here too, then, the subordinate clause expresses not only one thought but also part of another. Therefore one may not, in general, replace it by another of the same truth value.

49.2 It is difficult to exhaust all the possibilities that language offers; but I hope I have nonetheless found, in essentials, the reasons why a subordinate clause cannot always be replaced by another of the same truth value without detriment to the truth of the whole complex sentence. They are: 1) that the subordinate clause does not refer to a truth value, since it expresses only part of a thought; 2) that the subordinate clause does refer to a truth value but is not confined to this, since its sense comprises, besides one thought, part of another thought as well.

49.3 The first case arises a) with the indirect reference of the words, b) when a part of the sentence only indicates indefinitely instead of being a proper name.

49.4 In the second case the subordinate clause may have to be taken twice, namely once in its customary reference and once in its indirect reference; or the sense of a part of the subordinate clause may at the same time be a component of another thought, which, together with the thought directly expressed in the subordinate clause, makes up the whole sense of the main clause and subordinate clause.

49.5 From this it follows, with sufficient probability, that the cases where a subordinate clause cannot be replaced by another of the same truth value prove nothing against our view [50] that the truth value is the reference of a sentence whose sense is a thought.

BACK TO IDENTITY

The opening puzzle about a=a and a=b is resolved by the difference in sense.

50.1 Let us now return to our starting point.

50.2 If we found the cognitive value of "a=a" and "a=b" to be in general different, this is explained by the fact that, for cognitive value, the sense of the sentence—namely the thought expressed in it—is no less relevant than its reference, that is, its truth value. Now if a=b, then the reference of "b" is indeed the same as that of "a", and so the truth value of "a=b" is the same as that of

"a=a". Nevertheless, the sense of "b" may differ from the sense of "a", and therefore the thought expressed in "a=b" may also differ from that expressed in "a=a"; in that case the two sentences do not have the same cognitive value either. If, as above, we understand by "judgment" the advance from a thought to its truth value, we will also say that the judgments are different.